Master's thesis

Divisible Design Graphs

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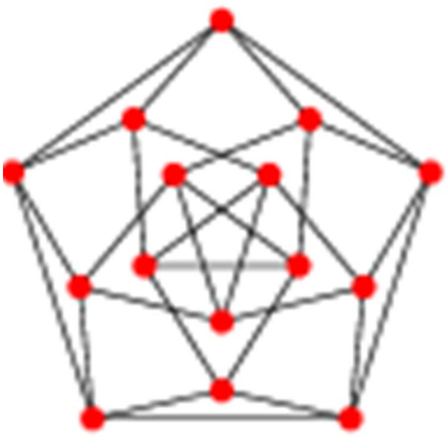

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Abstract

In this thesis we will describe a new class of graphs: divisible design graphs (DDG's) or more precise $(m, n, k, \lambda_1, \lambda_2)$-graphs. These are graphs that are also (group) divisible designs. First we derive a theoretical basis and conditions for the existence of a DDG. Furthermore the trivial cases are described: $\lambda_1 = k$ and $\lambda_2 = 0$ and the whole class of graphs with $\lambda_1 = k - 1$ can be classified. A computer search than found more possible parameter sets including the multiplicities of the eigenvalues. The results were divided into two classes: parameter sets with four or with five eigenvalues and some construction methods were found. This thesis includes a table with all possible parameters from the computer search on 50 vertices or less, with 241 open cases en 35 proven divisible design graphs; it is complete up to 20 vertices.

Preface

A defense of a Master's thesis should ideally mean that I could call myself virtually a Master of Science, although that is not true in my case, it sure feels like it. This Master's thesis is therefore the crown on what will probably be 5 and $\frac{1}{2}$ years of (hard) studying.
Studying has not always be fun, but I must admit that writing this thesis and doing the research has been kind of fun pretty much all the way. The best moments were when I could write down a new theorem, although it usually needed some alterations.

I could not have written this thesis without the support of Willem Haemers, whom I really need to thank for his support and inspiration for a subject for my thesis.
Finally I believe I have not moaned about "a boring thesis", but if anyone feels otherwise I sincerely apologize and thank him or her together with everyone else that is dear to me, for moral support or distraction.

The greatest adventure is what lies ahead.
Today and tomorrow are yet to be said.
(From: The greatest adventure,
Glenn Yarbrough (1977))

Maaike Meulenberg
Tilburg, September 29, 2008

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Chapter 1

Introduction

In this thesis we will investigate a new class of graphs: Divisible design graphs. As can be guessed from this title, it has to do both with divisible designs and graphs. We want to identify all graphs whose adjacency matrix is also a divisible design.
In other words we want to find graphs that allow for a division in groups of equal size such that any pair in the same group has λ_1 neighbors in common and any pair that is not from the same group has λ_2 neighbors in common.
The main goal of this thesis is to describe the divisible design of graphs, identify necessary conditions and come up with a list of parameter sets that may correspond to a divisible design graph. For the number of vertices smaller or equal to 20 we can identify all divisible design graphs by theorems or individually by other means.
The theorems that define divisible design graphs are in most cases defined with other graphs or combinatorial objects, therefore for large graphs, the theorems might not always be useful, since for large objects existence is not always verified.

In Chapter 2 and 3 we will first treat some concepts from graph and design theory, where the emphasis in Chapter 3 is on divisible design graphs. In Chapter 4 the divisible design graphs are formally defined, conditions and eigenvalues are given and theorems on a related matrix are proven. In Section 4.2 two trivial classes of divisible design graphs are described, these subclasses can both be described by one theorem and can therefore be excluded from the desired list of parameter sets.
In Chapter 5 the search for possible parameter sets is explained and started with for small number of vertices. In Section 5.2 all divisible design graphs with $\lambda_1 = k - 1$ are classified, which results in another subclass that can be easily described. In Section 5.3 and 5.4 we try to construct more graphs from the remaining parameter sets. In 5.3 this is done for parameter sets with 4 eigenvalues, in 5.4 for sets with 5 eigenvalues. Since more research has been done on graphs with four eigenvalues 5.3 is almost complete up to 30 vertices, 5.4 up to 20 vertices.
In Chapter 6 we will finish with some concluding remarks and recommendations for future research.

Chapter 2

Graphs

2.1 Preliminaries & Properties

A graph G is usually denoted as the set (V, E), where V is the collection of vertices and E the collection of edges, an edge is pair of vertices. If an edge ij exists point $i, j \in V$ are adjacent, if $i = j$ the edge is call a loop. In this thesis we will only consider undirected edges, i.e. if edge ij exists, edge ji is the same edge. A graph is *connected* if it is possible to 'walk' from i to j over the edges for any $i, j \in V$.
In every point one or more edges can come together, the number of edges that starts in point i is denoted as k_i. If $k_i = k$ for all $i \in V$, then the graph is called *k-regular*.
The sets of vertices and edges can be conveniently reported in an adjacency matrix A, where a_{ij} is 1 if there is an edge between i and j, 0 if not. This matrix is symmetric because the graph is undirected. If a graph has no loops the adjacency matrix has a zero diagonal, in this thesis this will always be the case.
The *complement* of a graph $\overline{G}$ has an adjacency matrix $J - A - I$, where J is an all one matrix, A the adjacency matrix and I the identity matrix.
A graph is *walk-regular* if the number of closed walks of length r is the same for every $v \in V$. A walk of length r between i and j exists if vertex i and j are connected through r edges (not necessarily distinct edges), if $i = j$ the walk is closed. So we have $(A^r)_{ij}$ is the number of walks of length r from i to j. We can check if a graph is walk-regular by looking at the diagonal of A^r, if the diagonal is constant for every r, the graph is walk-regular. In general it holds for a walk-regular graph that:

$$\theta_r = \frac{1}{v} \sum_{i=0}^{n} m_i \lambda_i^r \geq 0, \qquad \theta_r \in \mathbb{N} \tag{2.1}$$

Where λ_i is one of the n eigenvalues of the adjacency matrix and m_i the corresponding multiplicity.
The *diameter* d of a graph is defined as $d = \max_{i,j} c_{ij}$, where c_{ij} is the shortest path from i to j.
A graph can be *distance-regular*, which means that for any pair of vertices i and j and any $x, y \in \{0, \ldots d\}$ the number of vertices on distance x from i and distance y from j is constant for any i and j and depends solely on the choice of x and y and the distance between i and j.

2.2 The eigenvalues of the adjacency matrix

The eigenvalues of a graph are real numbers and there exist $|V|$ independent eigenvectors. This is true because any adjacency matrix is an integer valued symmetric matrix, thus the algebraic multiplicities are equal to the geometric multiplicities. The collection of eigenvalues with multiplicities is called the *spectrum* of a graph.
For a k-regular graph it holds that one of the eigenvalues is k and the corresponding multiplicity is 1 if the graph is connected, in general the multiplicity of k corresponds to the number of

components of the graphs (Haemers, 2007).
In a graph all eigenvalues must sum up to 0, since an adjacency matrix has a zero diagonal.

2.3 Special graphs

A *complete graph* on n vertices, has degree $n-1$ and therefore a total of $\frac{1}{2}n(n-1)$ edges. Every vertex is connected to every other vertex.
A *bipartite graph* is a graph (V, E) such that V can be split into two sets with no edges between members of the same set. We will denote a bipartite graph as $\mathrm{K}(m, n)$, where m is the size of a set and n the size of the other.
A *Strongly Regular Graph* (SRG) is defined by the parameters v, k, λ, μ. Where v is the number of vertices, k the degree of every point of the graph. λ is the number of neighbors two adjacent points have in common, whereas μ is the number of neighbors two non adjacent points have in common.
Furthermore a connected strongly regular graph has exactly three distinct eigenvalues and conversely any connected regular graph with precisely three distinct eigenvalues is a strongly regular graph (Haemers, 2007).
If a strongly regular graph has $\lambda = \mu$, then it is called a (v, k, λ)-graph. The adjacency matrix of a strongly regular graph with $\lambda = \mu$ corresponds to a 2-design.

A matrix is *circulant* if every row can be obtained from the row above, by moving every element one place to the right and the last entry goes the first position. A graph is circulant if it admits a circulant adjacency matrix.
A *cocktail party graph* is a graph with an even number of vertices v and any vertex is connected to all other vertices except one, therefore there are $\frac{1}{2}v$ pairs that are unconnected. The cocktail party graphs are called as such because the graph corresponds with a party where one shakes everybody's hand, except one's partner's hand.
A *line graph* $L(H)$ of a graph $H = (V, E)$ is a graph where every edge of H corresponds with a vertex in the line graph $L(H)$. In the line graph two vertices are connected if the edges they correspond with in H have a vertex $v \in V$ in common. The line graph has adjacency matrix $CC^T - 2I$ where I is the identity matrix en C is the matrix where every row denotes an edge of H, thus is of size $|E| \times v$.
A special line graph is the *triangular graph*, it is the line graph of a complete graph on v vertices and is denoted by $T(v)$, all triangular graphs are strongly regular.

Chapter 3

Design Theory

This Chapter provides the basics concepts in design theory, and especially divisible designs.

3.1 What is a design?

First we need to clearly define what a design is and what it is for. The field of block designs has its origin in the executing of agricultural experiments (Lint & Nienhuys, 1991). In these experiments a number of varieties were compared on their reaction to different treatments, e.g. different fertilizers. The varieties were put in blocks that each get the same treatment. A variety however can occur in more than one block, but within a block the varieties are treated identically. The field of block designs is now concerned with how to spread the varieties over the blocks: what is a 'good' way to do that, but also how many blocks do we need to get a 'good' result.
Formally we define two sets, a set $\mathscr{P}$ of points (varieties in the example) and a set $\mathscr{B}$ of blocks. Between those two sets we can define an incidence relation $\mathscr{D}$: a point $p \in \mathscr{P}$ is incident with a block $b \in \mathscr{B}$ if point p is in block b. The number of points is denoted as v and the number of blocks as b.

Example 3.1 An example of such an incidence relation can be made from a graph. The set $\mathscr{P}$ is the set of vertices of the graph, the set $\mathscr{B}$ is the set of edges. This means that every block is incident with only two points and the number of blocks is equal to the number of edges.

Note that the definition does not exclude repeated blocks, therefore there might be some blocks that are incident with exactly the same points. Repeated block are allowed, but we will see later that in the case of divisible design graphs, designs with repeated blocks, correspond to trivial designs.
The incidence matrix A of design $\mathscr{D}$ is a (0,1)-matrix of size $(v \times b)$in which:

$$(A)_{ij} = \begin{cases} 1 & \text{if point } i \in \mathscr{P} \text{ is incident with block } j \in \mathscr{B} \\ 0 & \text{if point } i \in \mathscr{P} \text{ is not incident with block } j \in \mathscr{B} \end{cases}$$

The general concept of an incidence structure is now defined, but it becomes interesting when it is somewhat regular, has some basic structural properties. We will start with defining a t-design.

Definition 3.1 An incidence structure with a set $\mathscr{P}$ of v points and a collection of blocks $\mathscr{B}$ is called a t-design with blocksize k and index λ if $v \geq k \geq t \geq 0$ and

1. Every block is incident with the same number of points, i.e. k points.
2. For every subset T of $\mathscr{P}$ with $|T| = t$ it holds that exactly λ blocks are incident with all points of T.

A t-design with parameters v, k, λ is denoted as a t-(v, k, λ) design. In all t-designs it holds that:

$$bk = rv \tag{3.1}$$

r is the replication number, i.e. the number of times a point appears in a block.
Note that any t-design is also al $t-1$-design. This is true because if all subsets of points size t are incident with λ blocks, then all subsets of any size smaller than t occur in a constant number of blocks as well.
Before we will introduce a special class of designs in section 3.2, the divisible designs, we will first go into detail for 2-designs and turn our attention to the square and/or symmetric designs.

3.1.1 2-designs

The 2-designs are a subclass of the t-designs. This class is especially important because a set of size two is a pair and looking at pairs makes sense in many cases. The definition of a 2-design is exactly the same as the t-design for $t = 2$, where λ is the number of times two points occur in the same block. A 2-design is sometimes called a Balanced Incomplete Block Design (BIBD).
The incidence matrix A of a 2-design has the following properties:

1. Every row of A has r times the number one.
2. Every column of A has k times the number one.
3. Two different rows have an inproduct of λ.

Example 3.2 A classic example of a 2-design is the Fano plane, which corresponds to a 2-(7,3,1) design. Block $B_0 = \{0, 1, 3\}$ and for every i with $1 \leq i \leq 6$ we find B_i by summing i to every element of B_0: $(B_0 + i) \mod 7$. The incidence matrix A can be found in Figure 3.1, note that the rows correspond to a point, the columns to a block.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

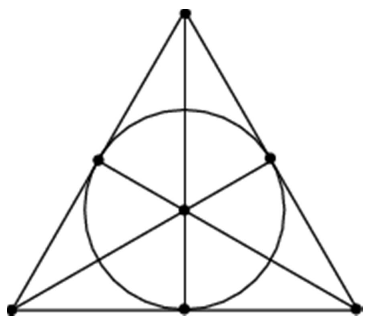


Figure 3.1: The Fano plane: incidence matrix (left) and the geometric representation (right)

In case of the Fano plane, the replication number r is 3 and so is k. It can also be seen that every treatment has exactly one block in common with any other treatment, which means every pair 'meets' each other exactly one time, $\lambda = 1$.
The Fano plane in the right of Figure 3.1 is drawn such that every block is represented by a line, the circle also represents one of the seven blocks.

Fisher's inequality holds for 2-designs and says that $b \geq v$. If it reduces to an equality, the number of blocks is equal to the number of vertices and we have a square design.

3.1.2 Square or symmetric?

Consider the case that the number of rows is equal to the number of columns in the design. In a such a square design $v = b$ and therefore also $r = k$, for notational convenience we will not use b

and r if the design matrix is square, only k and v. The Fano plane in Figure 3.1 for example is a square design.
Confusingly in literature square designs are sometimes called symmetric designs. Later on we will need to use the fact that a design matrix can be symmetric ($D = D^T$) therefore we will make a strict distinction between square and symmetric. Note that the Fano plane can have a symmetric design matrix[1], it can however not be made symmetric with zeros on the diagonal, which would make it correspond to a graph.

3.2 Divisible Designs

A divisible design (DD) is a special design, where the point set can be partitioned into m classes of size n, such that any two points in the same class occur together in λ_1 blocks and two points from different classes occur together in λ_2 blocks.
We know that:

$$v = mn \qquad \text{and} \qquad bk = vr \tag{3.2}$$

It also holds that:

$$r(k-1) = \lambda_1(n-1) + n(m-1)\lambda_2 \tag{3.3}$$

(3.3) is true because any point θ occurs in r blocks and all blocks contain $k-1$ points that θ can pair with. It must also hold that θ pairs λ_1 times with the $n-1$ members of his own class and λ_2 times with the $n(m-1)$ members of all other classes.
Combining (3.2) and (3.3) results in the fact that only five parameters are needed to describe a divisible design: DD$(m, n, k, \lambda_1, \lambda_2)$.
Since we are interested in proper divisible designs we will assume that $\lambda_1 \neq \lambda_2$, $m \geq 2$ and $n \geq 2$, otherwise this would be an ordinary block design. For the moment we will allow the use of repeated blocks, but in section 4.2 we will see that repeated blocks correspond with trivial cases.

We will now try do derive the eigenvalues of the incidence matrix A of a divisible design. AA^T gives us some structural information.

$$AA^T = \begin{pmatrix} C & B & \dots & B \\ B & C & \dots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \dots & C \end{pmatrix}$$

AA^T is built of two different block matrices C and B, both of size $(n \times n)$. Blocks of type C are on the diagonal of AA^T, blocks of type B are elsewhere, they look like:

$$C = \begin{pmatrix} r & \lambda_1 & \dots & \lambda_1 \\ \lambda_1 & r & \dots & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_1 & \dots & r \end{pmatrix} \qquad B = \begin{pmatrix} \lambda_2 & \lambda_2 & \dots & \lambda_2 \\ \lambda_2 & \lambda_2 & \dots & \lambda_2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_2 & \lambda_2 & \dots & \lambda_2 \end{pmatrix}$$

The form of C is easily explained by looking at the inproducts. The product of row i of A and column i of A^T is equal to r because they are identical and point i occurs in exactly r blocks. Off the diagonal of C we see λ_1, the inproduct of two rows corresponding to blocks of the same class. Matrix B consists completely of inproducts of blocks that are not of the same class and have an inproduct of λ_2. Therefore the elements of AA^T are defined by:

$$(AA^T)_{ab} = \sum_{j=1}^{v} (A)_{aj}(A)_{bj} = \begin{cases} r & \text{if } a = b \\ \lambda_1 & \text{if } a \text{ and } b \text{ belong to the same class} \\ \lambda_2 & \text{if } a \text{ and } b \text{ belong to different classes} \end{cases}$$

[1]Renumbering the blocks (or points) results in a symmetric matrix

Now we can write AA^T as the product of three matrices:

$$AA^T = rI_v + \lambda_1(K_{mn} - I_v) + \lambda_2(J_v - K_{mn}) \tag{3.4}$$

Where I_v represents the identity matrix of size v, J_v is a matrix $(v \times v)$ filled with ones and K_{mn} is defined as:

$$K_{mn} = I_m \otimes J_n = \begin{pmatrix} J_n & 0_n & \dots & 0_n \\ 0_n & J_n & \dots & 0_n \\ \vdots & \vdots & \ddots & \vdots \\ 0_n & 0_n & \dots & J_n \end{pmatrix}$$

Where 0_n is the $(n \times n)$ zero matrix. Since we now know that $AA^T \in \mathrm{span}\{K_{mn}, J_v, I_v\}$ we can find the eigenvalues of AA^T. To do that we need the eigenvalues and eigenvectors of K_{mn}, J_v and I_v.

- J_v has the eigenvalue v with multiplicity 1 and the eigenvalue 0 with multiplicity $v-1$. The vector 1_v is an eigenvector corresponding to the eigenvalue v.
- K_{mn} has the eigenvalue n with multiplicity m and the eigenvalue 0 with multiplicity $m(n-1)$. The vector 1_v is an eigenvector corresponding to the eigenvalue n.
- I_v has the eigenvalue 1 with multiplicity v.

AA^T is a symmetric integer matrix $(v \times v)$. Such a matrix has v real eigenvalues and v independent eigenvectors, that are orthogonal.
Suppose now we have some arbitrary combination $\alpha K_{mn} + \beta J_v$. To find eigenvalues of this new matrix we can sum the eigenvalues of K_{mn} and J_v if they correspond to the same eigenvector and obtain all eigenvalues of $\alpha K_{mn} + \beta J_v$. This is possible, because J_v and K_{mn} have v independent eigenvectors in common. In Table 3.1 all eigenvalues of the linear combination are given.

Eigenvector	Eigenvalues of $\alpha K_{mn} + \beta J_v$	Multiplicity
1_v	$\alpha n + \beta v$	1
an eigenvector of K_{mn} corresponding to n ($\perp 1_v$)	$\alpha n + \beta 0$	$m-1$
an eigenvector of K_{mn} corresponding to 0	$\alpha 0 + \beta 0$	$m(n-1)$

Table 3.1: Eigenvalues of $\alpha K_{mn} + \beta J_v$ with multiplicities

From now on we will call X the collection of eigenvectors of K_{mn} corresponding to the eigenvalue n but unequal to 1_v, and Y the collection of eigenvectors of K_{mn} corresponding to the eigenvalue 0.
We can find the eigenvalues of AA^T in the same way, by altering (3.4) somewhat into:

$$AA^T = (r - \lambda_1)I_v + \lambda_2 J_v + (\lambda_1 - \lambda_2)K_{mn} \tag{3.5}$$

Just as in the simple case of $\alpha K_{mn} + \beta J_v$ the eigenvalues of AA^T can be computed by summing the eigenvalues of the three matrices if they correspond to the same eigenvector. The eigenvalues of AA^T are in Table 3.2.

Eigenvector	Eigenvalues of AA^T	Multiplicity
1_v	$r - \lambda_1 + \lambda_2 v + (\lambda_1 - \lambda_2)n = rk$	1
$\forall\, x \in X$	$r - \lambda_1 + (\lambda_1 - \lambda_2)n = rk - \lambda_2 v$	$m-1$
$\forall\, y \in Y$	$r - \lambda_1$	$m(n-1)$

Table 3.2: Eigenvalues of AA^T with multiplicities

We can simplify the eigenvalue corresponding to 1_v by using expression (3.3) and rewrite that into:

$$rk = r - \lambda_1 + \lambda_2 v + (\lambda_1 - \lambda_2)n \tag{3.6}$$

With this equation the first and second eigenvalue can be simplified and in Table 3.2 the simplified eigenvalues are also presented.
AA^T is a positive semi-definite matrix[2] and therefore all eigenvalues are nonnegative. It is straightforward from the definition of a divisible design that $k - \lambda_1 \geq 0$, but also $rk - v\lambda_2 \geq 0$.
Based on this observation Bose & Conner (1952) define three classes of divisible designs, these classes are exhaustive and mutually exclusive:

- *Singular group divisible designs*, $r = \lambda_1$
- *Semi-regular group divisible designs*, $r > \lambda_1$ and $rk - \lambda_2 v = 0$
- *Regular group divisible designs*, $r > \lambda_1$ and $rk - \lambda_2 v > 0$

The restrictions the eigenvalues and multiplicities impose on the existence of a divisible design is one of the starting points of the search for the desired graphs. There are however more restrictions; we can use expression (3.3) and state some divisibility conditions. r needs to be an integer, thus dividing both sides of (3.3) with $k - 1$ must result in an integer. This results in the following corollary.

Corollary 3.2.1 *The following conditions are necessary for the existence of a divisible design:*

a. $(m-1)n\lambda_2 + (n-1)\lambda_1 \equiv 0 \mod k-1$

b. $v((m-1)n\lambda_2 + (n-1)\lambda_1) \equiv 0 \mod k(k-1)$

Proof. a. follows from (3.3) b. follows from (3.1) combined with (3.3). □

3.2.1 Dual Property

Some divisible designs possess a special property, the dual property. That means that not only the points can be partitioned in m classes of size n, with λ_1 and λ_2 as characteristic parameters, but the blocks can also be partitioned in m classes of size n on the basis of λ_1 and λ_2. The transpose of the matrix of such a divisible design is again a divisible design with the same characteristics.
A divisible design that possesses the dual property can therefore be partitioned in m^2 blocks of size $n \times n$. Bose (1977) has shown that within such a block the row and column sum are a constant r_{ij}. Bose puts these sums in a matrix R of size $(m \times m)$ and r_{ij} is the row/column sum of block B_{ij}. If in the remainder of this thesis B_{ij} is mentioned, we refer to a block within the design (or adjacency) matrix.

$$A = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1m} \\ B_{21} & B_{22} & \dots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \dots & B_{mm} \end{pmatrix}, R = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1m} \\ r_{21} & r_{22} & \dots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \dots & r_{mm} \end{pmatrix} \tag{3.7}$$

[2]It is a real valued symmetric matrix

Chapter 4

Divisible Design Graphs

In this Chapter the main topic of this thesis is introduced and defined. Furthermore some trivial constructions are described.

4.1 Definition

The subject of this thesis is Divisible Design Graphs (DDG's). These are graphs whose adjacency matrix is also the incidence matrix of a divisible design. If all parameters are known the DDG can be described as a (m,n,k,λ_1,λ_2)-graph. Clearly the divisible design corresponding to a divisible design graph possesses the dual property, because the matrix is symmetric.
So, if A is the incidence matrix of a divisible design, then A is the adjacency matrix of a divisible design graph whenever $A = A^T$ and A has a zero diagonal. Because A has k 1's per row (and column) any DDG will correspond to a k-regular graph
In Chapter 3.2 we already looked at the eigenvalues of AA^T for divisible designs in general. Since we now assume that A is symmetric it holds that $AA^T = A^2$ and since $r = k$ the eigenvalues of A^2 are now simplified and reported in Table 4.1.

Eigenvalues of A^2	Multiplicity
$\theta_0 = k^2$	1
$\theta_1 = k^2 - \lambda_2 v$	$m - 1$
$\theta_2 = k - \lambda_1$	$m(n - 1)$

Table 4.1: Eigenvalues of A^2 with multiplicities

Note that θ_1 and θ_2 cannot be equal:

$$\begin{aligned} k - \lambda_1 &= k^2 - \lambda_2 v \\ k - \lambda_1 &= \lambda_2 v + (\lambda_1 - \lambda_2)n + k - \lambda_1 - \lambda_2 v \\ 0 &= (\lambda_1 - \lambda_2)n \end{aligned}$$

In the second line of the equations above equation (3.6) is used. This equality results in the conclusion that $\lambda_1 = \lambda_2$, which cannot be the case, because then it would not be a proper divisible design.

The eigenvalues of A^2 in Table 4.1 give us much information of the eigenvalues of A, but do not determine them completely, the eigenvalues of A are in Table 4.2. The eigenvalues of A are the roots of those of A^2, but the multiplicities are not yet completely determined by the spectrum of A^2. These eigenvalues and multiplicities are interesting because they give information about the graph we want to find.

Eigenvalues of A	Multiplicity	
$\alpha_0 = k$	1	
$\alpha_1 = \sqrt{k-\lambda_1}$	f_1	
$\alpha_2 = -\sqrt{k-\lambda_1}$	f_2	$f_1 + f_2 = m(n-1)$
$\alpha_3 = \sqrt{k^2-\lambda_2 v}$	g_1	
$\alpha_4 = -\sqrt{k^2-\lambda_2 v}$	g_2	$g_1 + g_2 = m-1$

Table 4.2: Eigenvalues of A with algebraic multiplicities

Corollary 4.1.1 *A matrix A with more than 5 distinct eigenvalues cannot be a divisible design graph.*

Proof. The eigenvalues of A are the roots of the eigenvalues of A^2, k^2 has only multiplicity 1, thus A has k or $-k$. Since we know that a connected k-regular graph has one eigenvalue k, this is the one with multiplicity 1 and not $-k$. The other roots of the two eigenvalues of A^2 may be the positive as well as the negative root. □

The first condition that must hold is that the sum of all eigenvalues is equal to the trace of matrix, hence:

$$\text{trace } A = 0 = k + (f_1 - f_2)\sqrt{k-\lambda_1} + (g_1 - g_2)\sqrt{k^2 - \lambda_2 v} \tag{4.1}$$

Where f_1, f_2, g_1, g_2 are the multiplicities of eigenvalue α_1 through α_4. From (4.1) we can draw some conclusions:

- It is not possible that $f_1 = f_2$ and $g_1 = g_2$
- $k > 0$ therefore $f_2 > f_1$ and/or $g_2 > g_1$

The first conclusion is obvious because this would reduce (4.1) to $0 = k$ which is clearly a contradiction, because $k > 0$. Statement 2. holds because k is positive and therefore $(f_1 - f_2)$ or $(g_1 - g_2)$ must be negative. A not so obvious result from (4.1) is summarized in the following Theorem.

Theorem 4.1.1 *In a divisible design graph $k-\lambda_1$ or $k^2-\lambda_2 v$ is a square. If $k-\lambda_1$ is not a square, then $f_1 = f_2$ and if $k^2 - \lambda_2 v$ is not a square. then $g_1 = g_2$.*

Proof. Suppose both $k-\lambda_1$ and $k^2-\lambda_2 v$ are no proper squares , then the two roots multiplied by some integer must sum up to a proper integer. Let's call $\gamma_1 = f_1 - f_2$ and $\beta_1 = g_1 - g_2$:

$$-k = \gamma_1\sqrt{k-\lambda_1} + \beta_1\sqrt{k^2-\lambda_2 v} \tag{4.2}$$

Dividing $k-\lambda_1$ and $k^2-\lambda_2 v$ by square factors, we may write:

$$-k = \gamma_2\sqrt{q} + \beta_2\sqrt{p} \qquad \text{with } p \text{ and } q \text{ square free} \tag{4.3}$$

If we square the equation and because p and q square free it can be easily seen that $\sqrt{pq}$ must be an integer, so $q = p$. So the equation reduces to:

$$-k = (\gamma_2 + \beta_2)\sqrt{p} \tag{4.4}$$

For the equation to hold $\sqrt{p}$ must be rational and by construction also integer. This results in the conclusion that both $k-\lambda_1$ and $k^2-\lambda_2 v$ are proper squares, or only one of them combined with either γ_1 is zero or β_1 is zero, we already saw that they cannot both be zero. □

A necessary condition for the existence of a symmetrical regular divisible design with parameters $v, k, m, n, \lambda_1, \lambda_2$ is (Bose & Connor (1952)):

$$(k^2 - \lambda_2 v)^{m-1}(k-\lambda_1)^{m(n-1)} = a \quad \text{where } a \text{ is a square} \tag{4.5}$$

That is, the expression above is a perfect square. This condition holds for regular divisible designs ($k-\lambda_1 > 0$ and $k^2-\lambda_2 v > 0$). If either $k-\lambda_1$ or $k^2-\lambda_2 v$ is zero, the expression equals zero, which is also a square. If $k-\lambda_1 = 0$ then we have a trivial case, which is explained in 4.2.1. If $k^2-\lambda_2 v = 0$ then $k-\lambda_1$ will always be a square in case of a divisible design graph, since otherwise the trace cannot be zero.
This condition results in four situations:

1. The condition is automatically fulfilled if both $k^2-\lambda_2 v$ and $k-\lambda_1$ are squares.
2. If $k^2-\lambda_2 v$ is a non-zero square and $k-\lambda_1$ is not, m is even or n odd.
3. If $k-\lambda_1$ is a non-zero square and $k^2-\lambda_2 v$ is not, then m is odd.
4. If both $k^2-\lambda_2 v$ and $k-\lambda_1$ are not squares, both powers must be even, which means that m should be odd and n should also be odd.

Result 4. does hold in general, but we already know from (4.1) that this situation will not occur. The fact that the trace needs to be zero and $(k^2-\lambda_2 v)^{m-1}(k-\lambda_1)^{m(n-1)}$ is a square results in conditions on m and n or the multiplicities.

In the following theorem we refer to the matrix R, as defined in Bose (1977) to prove walk-regularity in a special case. Remember that the matrix R is a $(m \times m)$ matrix that has as entries the row sums of all the blocks of the design/adjacency matrix, see also 3.2.1.

Theorem 4.1.2 *If the matrix R of a divisible design graph has a constant diagonal, it is walk-regular.*

Proof. A is the adjacency matrix of the divisible design graph. It is clear that A^r has a constant diagonal if r is even, remember the form of A^2, for any even r it holds $A^r = aJ_v + bK + cI_v$. If $A^r \cdot A$ also has a constant diagonal, A^n always has a constant diagonal for any n. If R has a constant diagonal with entries d, the diagonal of $A^r \cdot A$ has a diagonal with entries $d(a+b)+(k-d)c$ on every place. If the diagonal of R is not constant the diagonal of $A^r \cdot A$ is not necessarily constant.□

Theorem 4.1.3 *The matrix R of a divisible design graph is symmetric and has the eigenvalues k, α_3, α_4 and corresponding multiplicities 1,g_1, g_2.*

Proof. The adjacency matrix A of a divisible design graph is always symmetric. Since this corresponds directly to a divisible design, the matrix R defined for divisible designs is also symmetric. From Bose (1977) we can easily see that the eigenvalues of R correspond to α_3 and α_4 of the divisible design graph.
The multiplicities of α_3 and α_4 are g_1 and g_2. Take an eigenvector w of R corresponding to α_3. We then have:

$$Rw = \alpha_3 w = \alpha_3 \begin{pmatrix} r_{11}w_1 + r_{12}w_2 + \cdots + r_{1m}w_m \\ \vdots \\ r_{m1}w_1 + r_{m2}w_2 + \cdots + r_{mm}w_m \end{pmatrix}$$

From this vector w we can construct an eigenvector w_A of the adjacency matrix A of the DDG, this vector has the element w_1 on the first n positions, w_2 on the next n positions and so on. We now have:

$$Aw_A = \alpha_3 \begin{pmatrix} r_{11}w_1 + r_{12}w_2 + \cdots + r_{1m}w_m \\ \vdots \\ r_{11}w_1 + r_{12}w_2 + \cdots + r_{1m}w_m \\ \vdots \\ r_{m1}w_1 + r_{m2}w_2 + \cdots + r_{mm}w_m \\ \vdots \\ r_{m1}w_1 + r_{m2}w_2 + \cdots + r_{mm}w_m \end{pmatrix}$$

Thus, from every eigenvector of R corresponding to α_3 we can make an eigenvector of A, for the eigenvectors of α_4 holds the same. The multiplicities of the eigenvalues α_3 and α_4 of R and A can only be exactly the same, that is g_1 and g_2.
Note that this reasoning only holds because α_1 and α_2 cannot be equal to α_3 and α_4, which is true in case of divisible design graphs. □

Corollary 4.1.2 *The matrix R of a divisible design graph has $0 \leq D \leq D_{max}$, where D represents the trace of R and $D_{max} = (n-1)m$.*

Proof. In a matrix the trace is equal to the some of the eigenvalues, D in this case, thus $D = k + (g_1 - g_2)\alpha_3$. Since a divisible design graph has only positive entries, the trace cannot be smaller than zero. On the diagonal there can not be blocks J_n, since a graph has a zero diagonal, so the maximal trace can be $(n-1)m$. □

4.2 Trivial cases of DDG

Before checking all possible parameter sets to find the desired divisible design graphs, we will exclude two trivial cases.

4.2.1 $\lambda_2 = 0$

We will start this section with giving an example of a DDG with $\lambda_2 = 0$ and then proceed to generalize the example to prove that this generalization is the only possibility when $\lambda_2 = 0$.

Example 4.1 The picture in Figure 4.1 is clearly a (2,4,3,2,0)-graph, a divisible design graph. The graph can be split into two groups of size four each, the upper and lower row of points in the picture. The inproducts of the rows of the same groups is $\lambda_1 = 2$ and those of rows of different groups $\lambda_2 = 0$. The graph itself is a bipartite graph and can for example be found by drawing the incidence graph of a 2-(4,3,2)-design.
An incidence graph means that the blocks of the 2-(v,k,λ)-design are drawn as vertices in the graph together with the points of the design. An edge is drawn if point i is in block j. This results in a k-regular graph, that is also a divisible design graph with $\lambda_1 = \lambda$ and $\lambda_2 = 0$, no edges are drawn between two points or two blocks.

$$A = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

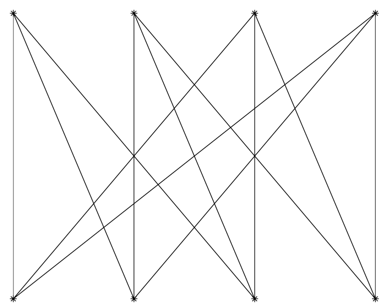


Figure 4.1: Example of a divisible design graph with $\lambda_2 = 0$

The matrix A in Figure 4.1 is built of four blocks. On the diagonal we have two zero matrices of size four. In the upper right corner there is a 2-(4,3,2) design and in the lower right corner is the transpose of the incidence matrix of this design. Note that the example in Figure 4.1 is the cube graph.

We will now show that every DDG with $\lambda_2 = 0$ is always built of a specific kind of blocks.

Theorem 4.2.1 *A divisible design graph with $\lambda_2 = 0$ (and $\lambda_1 > 0$) has adjacency matrix A with exactly one non-zero block in every row and column. A non-zero block on the diagonal is a (n,k,λ_1)-graph, other non-zero blocks are square 2-(n,k,λ_1) designs with $B_{ji} = (B_{ij})^T$.*

Proof. It is clear that if A is built as the theorem prescribes, we have a DDG with $\lambda_2 = 0$. We will now show that this is the only way of constructing a DDG with $\lambda_2 = 0$. So we will construct a matrix A:

$$A = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1m} \\ B_{21} & B_{22} & \dots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \dots & B_{mm} \end{pmatrix}, B_{ij} = B_{ji} \in \mathbb{R}^{n \times n} \tag{4.6}$$

Suppose there is an element $b_{xy} = 1$ in a certain block $B_{ij}, i \neq j$ and we want to construct a DDG with $\lambda_2 = 0$. Assume:

$$b_{xy} = 1, \quad b_{xy} \in B_{ij}, \quad x, y \in \{1 \dots v\}$$

Then row x will consist of only zeros except in block B_{ij}, for column y, row y and column x holds the same. So in row x you still need to place $k-1$ ones, they can only be placed within block B_{ij}. Placing these ones results in more columns that are for the largest part filled with zeros (and because of symmetry, also lots of rows).
But since A must also correspond to a divisible design, the row x must have an inproduct of λ_1 with the rows from the group it belongs to. Since it must hold for every row of the group, every row in the block B_{ij} will get at least one 1, since λ_1 is strictly larger than 0. This implies that $B_{ik} = 0_n$ if $k \neq j$. The only way the ones within B_{ij} can be be placed is as a square 2-(n,k,λ_1) design. Because of symmetry, column j has been determined also.
The blocks that are yet undetermined can be filled the same way which means if there is a 1 in any off-diagonal block that is not yet filled, we will have the same situation as before and more zero blocks and two 2-designs will. If there is a one in a diagonal block B_{ii} this specific block must be a (n,k,λ_1)-graph and any other block in row i and column i are zero blocks. □

Corollary 4.2.1 *If adjacency matrix A corresponds to a connected divisible design graph with $\lambda_2 = 0$, then $m = 2$.*

Proof. Follows directly from the proof of Theorem 4.2.1. □

The eigenvalues of a connected divisible design graph with $\lambda_2 = 0$ are in Table 4.3. Since it is connected α_3 has multiplicity 0 and α_4 multiplicity $m - 1 = 1$. The only way the trace can now be zero is when α_1 and α_2 have the same multiplicity $n - 1$.

Eigenvalues of A (4.2.1)	Multiplicity
$\alpha_0 = k$	1
$\alpha_1 = \sqrt{k - \lambda_1}$	$n-1$
$\alpha_2 = -\sqrt{k - \lambda_1}$	$n-1$
$\alpha_4 = -k$	1

Table 4.3: Eigenvalues of connected divisible design graphs with $\lambda_2 = 0$

Corollary 4.2.2 *Every connected divisible design graph with $\lambda_2 = 0$ (and $m = 2$) corresponds to a connected bipartite k-regular graph.*

Proof. Follows directly from the proof of Theorem 4.2.1. Note that the reverse does not hold, take for example the k-regular bipartite graph in Figure 4.2, the circular graph C_8. □

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

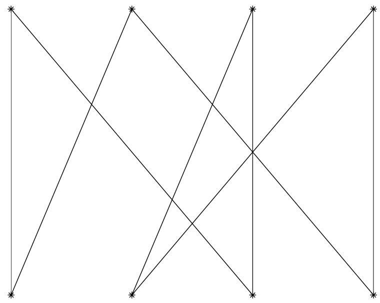


Figure 4.2: Example of a bipartite k-regular graph that is not a divisible design graph

Corollary 4.2.3 *A connected divisible design graph with $\lambda_2 = 0$ is the incidence graph of a symmetric 2-design.*

Proof. A line is drawn in the graph from vertex x to y if the point, vertex x is representing, is in the block that y represents. Which means that only off-diagonal we will encounter non-zero entries. Any connected divisible design graph has only two groups, therefore $\lambda_2 = 0$. We can also show that λ_1 is constant, because the off-diagonal blocks correspond exactly to the original 2-design, which has a constant λ, thus $\lambda = \lambda_1$. □

4.2.2 $\lambda_1 = k$ or $\lambda_2 = k$

We also have a trivial case when $\lambda_1 = k$. This situation can only occur if the rows of the matrix A are the same within a group. Which makes the design itself already less interesting to study, quite often definitions exclude these repeated blocks.

Theorem 4.2.2 *A (connected) divisible design graph with $\lambda_1 = k$ (and $\lambda_2 > 0$) exists if and only if there exists a $(m, \frac{k}{n}, \frac{\lambda_2}{n})$-graph.*

Proof. Note again that all rows of the same group must be identical to have an inproduct of k. This means that all blocks on the diagonal $B_{ii} = 0_n$, because a graph has a zero diagonal. Because all rows/columns in a group are equal, the blocks $B_{ij}, i \neq j$ can only be of the type J_n or 0_n.
It is easily verified that if there exists a (v, k, λ)-graph with the desired parameters and adjacency matrix A, $A \otimes J_n$ is a divisible design graph with $\lambda_1 = k$. We will now show this is the only possibility for such a graph to exist.
Since the blocks B_{ij} are either J_n or 0_n, it is enough to find a symmetric (0,1)-matrix C $(m \times m)$ with zeros on the diagonal, (thus C corresponds to a graph) that satisfies $C \otimes J_n$ to be a divisible design graph. In order to find a divisible design graph C must have $\frac{k}{n}$ 1's per row (and column). The inproduct of the rows of C is multiplied with n in the Kronecker product and therefore corresponds to λ_2, so if the inproduct of C is equal to $\frac{\lambda_2}{n}$ the inproduct of $C \otimes J_n$ has exactly the desired inproducts. If this inproduct is not equal within C, λ_2 will also never be constant. The C we have now constructed is exactly the definition of a (v, k, λ)-graph. □

If $C = J_2 - I_2$, $C \otimes J_n$ is a complete bipartite graph.

Corollary 4.2.4 *A connected divisible design graph with $\lambda_1 = k$ exists only if k and λ_2 are divisible by n.*

Proof. Follows immediately from the proof of Theorem 4.2.2. □

The eigenvalues of a connected divisible design graph with $\lambda_1 = k$ are in Table 4.4, the multiplicities are found with Haemers (2007).

Eigenvalues of A (4.2.2)	Multiplicity
$\alpha_0 = k$	1
$\alpha_1 = \alpha_2 = 0$	$m(n-1)$
$\alpha_3 = \sqrt{k^2 - \lambda_2 v}$	$\frac{1}{2}\left(m - 1 - \frac{k}{\sqrt{k^2-\lambda_2 v}}\right)$
$\alpha_4 = -\sqrt{k^2 - \lambda_2 v}$	$\frac{1}{2}\left(m - 1 + \frac{k}{\sqrt{k^2-\lambda_2 v}}\right)$

Table 4.4: Eigenvalues of connected divisible design graphs with $\lambda_1 = k$

Theorem 4.2.3 *A divisible design graph with $\lambda_2 = k$ does not exist.*

Proof. For two rows to have an inproduct of k the rows must be identical. This means that all rows of the adjacency matrix A must be identical, which can never correspond to a (connected) graph. □

Chapter 5

Non-trivial divisible design graphs

In this chapter non-trivial divisible design graphs will be searched for, in contrast with the trivial graphs of the previous chapter. First the procedure that selects possible parameter sets for divisible design graphs is explained, which uses definitions or conclusions from Chapter 3 and 4. We then show all divisible design graphs on 10 vertices or less. Then we will uncover a special class of non-trivial designs, divisible design graphs with $\lambda_1 = k - 1$.
The last two sections are dedicated to divisible design graphs with 4 or 5 eigenvalues respectively.

5.1 Searching for possible parameter sets

The chapters on designs and divisible design graphs already show the most important restrictions we can impose on the search. At first the necessary condition (4.5) is important and of course the trace of the graph must be zero and equal to the sum of the eigenvalues (4.1), but also the more general necessary condition from Corollary 3.2.1 must be true. The remaining options must be such that they can potentially correspond to both a graph and a divisble design.
The procedure we used to search for a divisible design graph starts with a given number of points v and proceeds as follows:

Step 0. Checks whether v is not a prime. Since we have defined a group divisible design as having groups of equal size, v cannot be prime.

Step 1. All possible number of groups m and group sizes n are determined with help of the prime factorization of v. The number of combinations is equal to the proper number of divisors of v.

Step 2. For all combinations of k, λ_1, λ_2 and m (with corresponding n) it is checked whether (4.5) is a proper square and whether $k^2 - \lambda_2 v \geq 0$. This is done with the use of the following restrictions:

- k runs from 3 to $v-3$. A degree of 1 will not correspond to a connected graph, a degree of 2 gives a cycle graph or a disjoint union of cycle graphs and will not correspond to a non-trivial divisible design. The large degrees $v-1$ and $v-2$ are excluded because complete graphs or cocktail party graphs cannot correspond to a divisible design graph.
- It cannot be so that v is odd and k is odd.
- Both λ_1 and λ_2 run from a to $k-1$, where $a = \max(0, 2k - v)$. This lower bound is as such because the inproduct of two arbitrary rows of the adjacency matrix of a k-regular graph is 0 or $2k - v$ in the 'worst' case. The intuition behind this is, that if k is relatively large, λ_i cannot be small. The upper bound for λ_1 is as such because the trivial case is excluded and for λ_2 because it cannot exist.
- Exclusion of the cases for which $\lambda_1 = \lambda_2$, $\lambda_2 = 0$.

Step 3. The necessary conditions from Corollary 3.2.1 are checked and it is checked whether (3.6) holds.

Step 4. For the remaining possibilities it is checked whether $k^2 - \lambda_2 v$ and/or $k - \lambda_1$ are squares. If they are not, the corresponding multiplicities are equal, e.g. if $k^2 - \lambda_2 v$ is not a square $g_1 = g_2 = \frac{(m-1)}{2}$.

Step 5. In this step the multiplicities are looked at more carefully:

- Options for which $f_1 = f_2$ and $g_1 = g_2$ are excluded
- Options for which $f_1 = f_2$ or $g_1 = g_2$ are not integers are excluded.
- Options where $\lambda_1 = k - 1$ and n is odd are discarded, in section 5.2 we will see why.
- Options with $k^2 - \lambda_2 v = k - \lambda_1$ are discarded. Because in Chapter 4 we already saw this cannot occur.
- If only one of $k^2 - \lambda_2 v$ or $k - \lambda_1$ is a not a square, the yet undetermined multiplicities can be computed and checked if they are integers.
- If both $k^2 - \lambda_2 v$ and $k - \lambda_1$ are squares, the multiplicities are not yet determined and remain in the list.

Step 6. If both $k^2 - \lambda_2 v$ and $k - \lambda_1$ are squares, the multiplicities can be found by checking all combinations. Every combinations is checked for (4.1). If that holds it is an option and it is stored, if the equality does not hold the option is discarded. The options for which $k^2 - \lambda_2 v = 0$ are automatically assigned $g_1 = m - 1$ and $g_2 = 0$, since different multiplicities do not matter if the eigenvalue is identical, zero in this case.

Step 7. Options for which only three distinct eigenvalues remain are excluded, i.e. f_1 or f_2 and g_1 or g_2 are zero. A connected simple and k-regular graph with three distinct eigenvalues is a strongly regular graph, never a divisible design graph.

The procedure above results in the options for $v \leq 15$ in Table 5.1. In this table the options with $\lambda_1 = k - 1$ are also mentioned separately because they can be classified on their own, see section 5.2.

v	Total number of options	Options with $\lambda_1 = k - 1$
4	0	0
6	1	1
8	3	1
9	0	0
10	1	1
12	11	5
14	2	1
15	3	0

Table 5.1: Options $v \leq 15$

The figures in Table 5.1 include the options with the same parameters, but only the multiplicities vary.

5.1.1 Checking for $n \leq 10$

As one can imagine, the number of options that might describe a divisible design graph grows rapidly if v grows. For small v we can check quite easily whether a certain parameter set corresponds to a k-regular graph, because these graphs are known and their adjacency matrices can be used to check the options. In this section the remaining options will be checked for $v = 6, 8, 10,$

then n becomes too large and we will have to use other methods. $v = 4, 9$ do not get any more attention, since it has already been tested that there do not exist non-trivial divisible design graphs on 4 or 9 vertices.
In this section we will also look at divisible design graphs with $\lambda_1 = k - 1$, which are classified on their own in section 5.2.
In the following sections all k-regular graphs have been reported by Meringer (2004).

Divisible design graphs for $v = 6$

The one remaining parameter set for a divisible design graph on 6 vertices is summarized in Table 5.2.

v	k	λ_1	λ_2	m	n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$	DDG?
6	3	2	1	3	2	-	-1^3	$\sqrt{3}^1$	$-\sqrt{3}^1$	No

Table 5.2: Option for a GDD when $v = 6$

If there exists a divisible design graph on 6 vertices it must be on of the two graphs in Figure 5.1. Because those are the only two 3-regular graphs on 6 vertices.

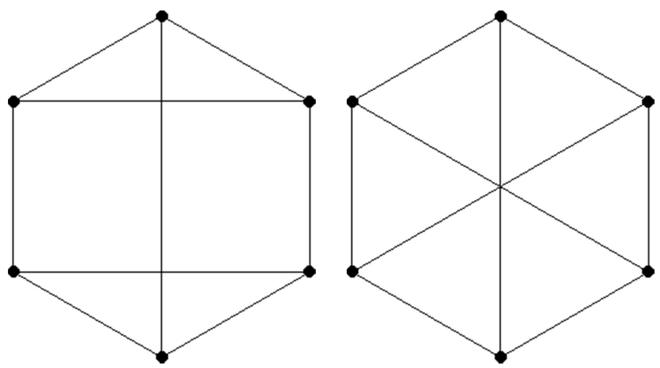


Figure 5.1: All 3-regular graphs on 6 vertices

The adjacency matrix of the left graph in Figure 5.1 has eigenvalues and multiplicities $(-2^2, 0^2, 1^1, 3^1)$, the eigenvalues and multiplicities do not correspond, hence the left graph is no divisible design graph. The right graph has eigenvalues and multiplicities $(-3, 0^4, 3)$. The right graph is also not a divisible design graph.
This leads to the conclusion that there is no non-trivial divisible design graph on 6 vertices.

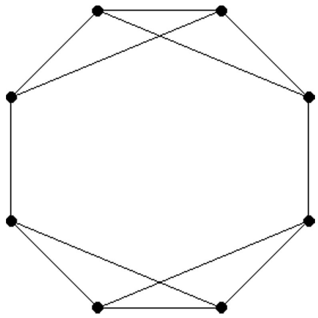
Divisible design graphs for $v = 8$

When we want to find a non-trivial divisible design graph with 8 vertices there are only 3 possibilities to check, see Table 5.3, only one of these has non-integer eigenvalues (1).

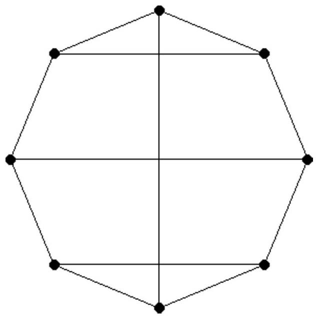
Index	v	k	λ_1	λ_2	m	n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$	DDG?
1	8	3	0	1	4	2	$\sqrt{3}^2$	$-\sqrt{3}^2$	-	-1^3	No
2	8	4	0	2	4	2	2^1	-2^3	0^3	-	Yes
3	8	5	4	2	2	4	1^2	-1^4	-	-3^1	Yes

Table 5.3: Possible parameter sets of a DDG on 8 vertices

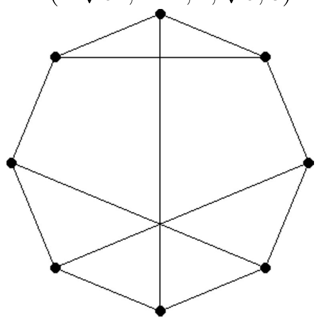
In Figure 5.2 all possible 3-regular graphs can be found. If option 1 in Table 5.3 corresponds to a divisible design graph, it must correspond to one of those graphs. Comparing the eigenvalues and multiplicities in Table 5.3 with those in Figure 5.2 tells us that parameter set 1 in Table 5.3 does *not* correspond to a divisible design graph.



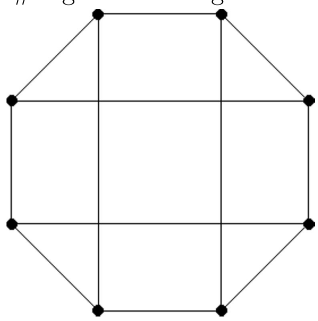
1:$(-\sqrt{5}^1, -1^4, 1, \sqrt{5}, 3)$



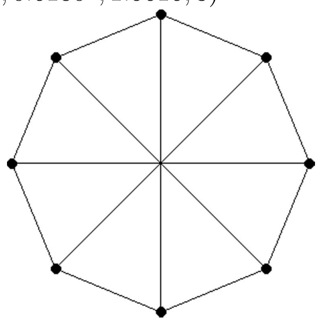
2:# eigenvalues larger than 5



3:$(-2.5616, -1.6180^2, 0, 0.6180^2, 1.5616, 3)$



4:$(-3^1, -1^3, 1^3, 3)$



5:$(-1-\sqrt{2}^2, -1, -1+\sqrt{2}^2, 1^2, 3)$

Figure 5.2: All possible 3-regular graphs on 8 vertices with corresponding eigenvalues

Since there does not exist a (non-trivial) 3-regular divisible design graph, we will go on to the check whether there are 4-regular graphs. Option 2 in Table 5.3 with $k = 4$ has $k^2 - \lambda_2 v = 0$ and therefore precisely 4 eigenvalues. There exist 6 different 4-regular graphs, in Figure 5.3 all 4-regular graphs are given with their eigenvalues.
The parameter set has 4 distinct eigenvalues, therefore only the graph in the upper right corner of Figure 5.3 might be a divisible design graph. The eigenvalues and multiplicities also correspond exactly with those in Table 5.3 and index 2, therefore it might be a (4,2,4,0,2)-graph.

When checking all possibilities we have never been certain that the parameters indeed correspond to a divisible design graph, so before we can say this is a divisible design graph, the graph must form a divisible design. This can be checked by taking the adjacency matrix[1] of the graph to

[1] The adjacency matrix is changed such that the rows and columns of the same groups are put next to each other.

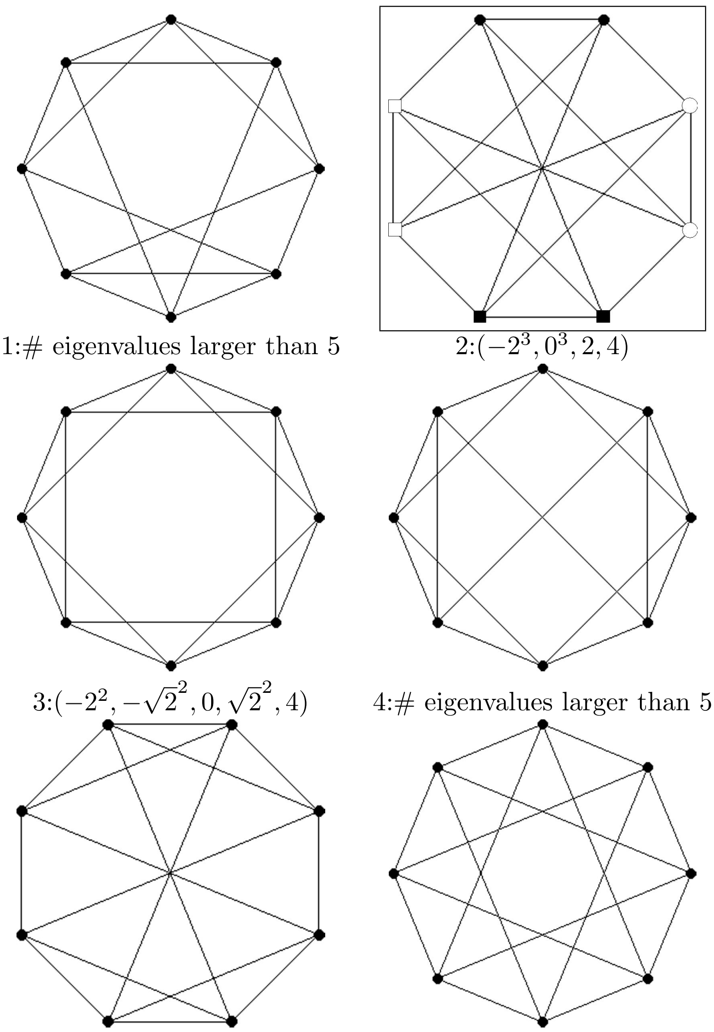
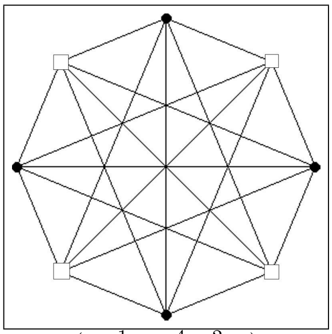
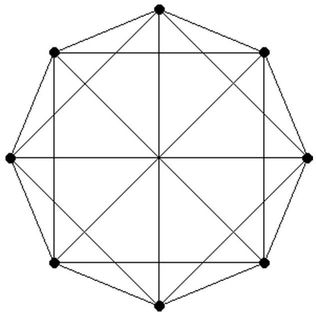


Figure 5.3: All possible 4-regular graphs on 8 vertices with corresponding eigenvalues

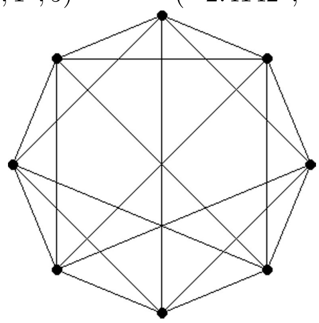
the power 2, if this corresponds to equation (3.4) then the adjacency matrix of the graph also corresponds to a divisible design graph.
And indeed option 2 corresponds to a divisible design graph that looks like graph 2 in Figure 5.3. The points that are in the same group are the points that are connected with a strict vertical or horizontal line in the picture. The graph is the complement of the cube graph.
The last option on 8 vertices is for $k = 5$ and might correspond to a 5-regular graph. In Figure 5.4 the possible graphs are drawn and the last parameter set is option 3 in Table 5.3.



$(-3^1, -1^4, 1^2, 5)$



$(-2.4142^2, -1^2, 0.4142^2, 1, 5)$



$(-3^1, -1.6180^2, 0^2, 0.6180^2, 5)$

Figure 5.4: All possible 5-regular graphs on 8 vertices with corresponding eigenvalues

Just by looking at the eigenvalues in Figure 5.4 we already know that graph 2 and 3 will never correspond to option 3 in Table 5.3, because the eigenvalues of these graph in the Figure are not all integers, whereas option 3 has integer eigenvalues. Comparing the multiplicities and eigenvalues results in the possibility that graph 1 in Figure 5.4 corresponds to option 3. The last check whether this is really a divisible design graph is positive and therefore we have found the second and last divisible design graph on 8 vertices. The points are alternating in group 1 or 2. This graph is called a circulant graph on 8 vertices: $C_8(1, 3, 4)$.

Divisible design graphs for $v = 10$

v	k	λ_1	λ_2	m	n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$	DDG?
10	5	4	2	5	2	-	-1^5	$\sqrt{5}^2$	$-\sqrt{5}^2$	Yes

Table 5.4: Possible parameter set for a DDG on 10 vertices

In Table 5.4 we see that there is only one possible parameter set left with $k = 5$. There are 60 5-regular graphs on 10 vertices. We will check whether one of the 60 5-regular graphs corresponds

If this is not possible without sacrificing symmetry or a zero trace, the matrix cannot correspond to a divisible design graph

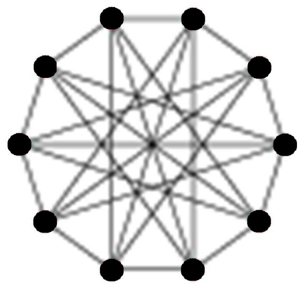


Figure 5.5: A DDG on 10 vertices with $k = 5$

to the parameters in Table 5.4 and indeed there turns out to be a divisible design graph. The graph is drawn in Figure 5.5, where two opposite vertices are in the same group. This specific graph is called a circulant graph on 10 vertices: $C_{10}(1, 4, 5)$.

5.2 Divisible designs with $\lambda_1 = k - 1$

In the previous section we saw some non-trivial designs for small v. It is however not possible to check the possibilities for large v in the same way. A number of options have $\lambda_1 = k - 1$, in Table 5.1 it can be seen the fraction is substantial. In Haemers (1991) these kind of divisible designs are classified and it is proven how the incidence matrix should look.

Theorem 5.2.1 (Haemers, 1991) *An incidence structure $\mathscr{D}$ is a divisible design with $r - \lambda_1 = 1$ if and only if $\mathscr{D}$ or the complement of $\mathscr{D}$ has an incidence matrix $(A \otimes J) + I$, where one of the following holds:*

(i) $J - 2A$ is the core of a skew-symmetric Hadamard matrix.

(ii) J has size (2×2) and A is the adjacency matrix of a strongly regular graph with $\mu - \lambda = 1$.

(iii) $A = 0$ or $A = J - I$.

This means that any divisible designs with $\lambda_1 = k-1$ falls in one of the classes above. Class (i) does not have a symmetric incidence matrix, therefore will never correspond to a DDG. Furthermore we can conclude from this that a DDG on an odd number of points cannot occur.

Theorem 5.2.2 *A divisible design graph with $\lambda_1 = k - 1$ has an even number of points v.*

Proof. A divisible design graph cannot fall in class (i), therefore it must be in class (ii) or (iii). It is straightforward that it can only fall in class (ii) if the number of points is even. The complement of class (ii) cases is a situation where $\lambda_1 = k$.
In class (iii), there are two types $A = 0$ and $A = J - I$. If $A = 0$ the incidence matrix is I or its complement $J - I$ which are both no strict divisible design and do not correspond to a divisible design graph. This leaves $A = J - I$, in this case the incidence matrix has blocks J off the diagonal and blocks I on the diagonal. If you interchange row 1 and 2, 3 and 4 and so on. There will arise a zero diagonal and the matrix is still symmetric and corresponds to a divisible design and therefore also a divisible design graph. This interchanging of the rows is only possible when v is even, if it is odd, the group size n is also odd and there will be no possibility to interchange the last row of every group such that there is a zero diagonal and the matrix remains symmetric.
The only case left to describe the complement of $(A \otimes J) + I$ when $A = J - I$. If this is true, then $\lambda_2 = 0$, which has already been described as a trivial case, see Theorem 4.2.1, but also $\lambda_1 = k - 2$ □

Corollary 5.2.1 *A divisible design graph with $\lambda_1 = k - 1$ has an even group size n.*

Proof. This follows with the same argument as in Theorem 5.2.2. The divisible designs in class (ii) always have an even group size. If the group size in class (iii) is odd, the incidence matrix cannot be kept symmetric and with a zero diagonal. □

This last corollary is already implemented in the search procedure in 5.1.
All options of class (iii) that correspond with a divisible design graph are summarized in the following Theorem.

Theorem 5.2.3 *If* $v \equiv 0 \mod a, \quad a \in \{4, 6, 8, \dots\}$ *and* $v \neq a$. *Then there exists a divisible design graph with parameters* $(\frac{a}{2}, \frac{2v}{a}, v - \frac{2v}{a} + 1, v - \frac{2v}{a}, v - 2\frac{2v}{a} + 2)$.

Proof. We will show that if v is as such it is a divisible design of class (iii) and corresponds to a graph because it is symmetric with a non zero diagonal. Take the number of groups equal to $m = \frac{a}{2}$ which is possible because a is even. Then $n = \frac{2v}{a}$. The divisible design graph is in class (iii) not because $A = 0$, but because $A = J - I$. Take $A = J_m - I_m$. $(A \otimes J_n) + I_v$ is now equal to a matrix with ones, except for the blocks on the diagonal those are identities. Interchanging row 1 and 2, 3 and 4 and so on results in the desired graph, where k is equal to $(m-1)n + 1 = v - n + 1$ because there are in every row $m - 1$ blocks with 1's and a one extra. Within a group it is clear that $\lambda_1 = k - 1$ because the rows are identical except for the 'extra' one. The inproduct of two rows of different groups is equal to $\lambda_2 = (m-2)n + 2 = v - 2n + 2$. Which proofs the existence of such a divisible design graph, since the incidence matrix is still symmetric and with a zero diagonal. □

One of the divisible design graphs defined by Theorem 5.2.3 is for example the graph on 8 vertices with index 3 in Table 5.3. It is not possible to have divisible design graphs defined by the theorem where $a = 2$, because in that case we will end up with the identity matrix or a complete graph (complement of $A \otimes J) + I$), not a divisible design graph. This is also the reason why other even numbers like 10, 14 for example , cannot be a divisible design graph of this type, because they are divisible by 2 and some odd number, both of which cannot be equal to n, $n = 2$ results in an complete graph and odd n is not possible, see also Theorem 5.2.1. For $v \leq 50$ there are 38 divisible design graphs[2] defined by Theorem 5.2.3.
The eigenvalues and multiplicities can be determined just as in Chapter 3. For divisible design graphs, defined by Theorem 5.2.3, it holds that the adjacency matrix A equals:

$$A = J_v - K_{mn} + B, \qquad \text{where} \quad B = I_{\frac{1}{2}v} \otimes (J_2 - I_2) \tag{5.1}$$

In section 3.2 $J_v, K_{mn}, 0_{\frac{1}{2}v}$ are defined, the eigenvalues of them are also explained there. The eigenvalues of B are -1 with multiplicity $\frac{1}{2}v$ and 1 also with multiplicity $\frac{1}{2}v$. The eigenvalues of divisible designs defined by Theorem 5.2.3 are in Table 5.5.

Eigenvector	Eigenvalues of A (5.2.3)	Multiplicity
1_v	$v - n + 1 = k$	1
an eigenvector of K_{mn} corresponding to n (not 1_v)	$-n + 1$	$m - 1$
an eigenvector of B, not of K_{mn} corresponding to 1	1	$\frac{1}{2}v - m$
an eigenvector of B, corresponding to -1	-1	$\frac{1}{2}v$

Table 5.5: Eigenvalues of the divisible design graphs defined by Theorem 5.2.3

Other divisible design graphs with $\lambda_1 = k - 1$ can now only be divisible designs of class (ii) defined by Haemers. So for such a divisible design graph to exist there must exist a strongly regular graph with specific parameters.

Theorem 5.2.4 *If a strongly regular graph on* v *vertices and valency* k *with* $\mu - \lambda = 1$ *exists, there exists an* $(v, 2, 2k + 1, 2k, 2\lambda + 2)$*-graph.*

[2] These are *not* reported in Table B.1

Proof. It follows directly from Theorem 5.2.1 that there exists a divisible design made with the strongly regular graph. The number of groups is v because every 1 in the adjacency matrix of the SRG is replaced by a (2×2) matrix of 1's and n is 2. the degree of the divisible design is equal to $2k+1$. The inproduct within a row is exactly $2k$, only the 'extra' ones dot not correspond. λ_2 is equal to $2\mu = 2\lambda + 2$.
Furthermore this construction from a SRG corresponds to a DDG, because it is symmetric, since the SRG is symmetric and the zero diagonal can be found by interchanging row 1 and 2, 3 and 4 and so on. □

To find all divisible design graphs defined by Theorem 5.2.4 on 50 vertices or less, we need to find all strongly regular graphs on 25 vertices or less, that have $\mu - \lambda = 1$. There exist 7 of such strongly regular graphs on 25 vertices or less, see Table 5.6. The parameters are reported by Brouwer (n.d.). When a parameter set corresponds to more than one strongly regular graph, they all correspond to different divisible design graphs.

v	k	λ	μ	# SRG's
5	2	0	1	1
9	4	1	2	1
10	3	0	1	1
10	6	3	4	1
13	6	2	3	1
17	8	3	4	1
25	12	5	6	15

Table 5.6: Strongly Regular Graphs with $\mu - \lambda = 1$ and $v \leq 25$.

From Table 5.6 we can conclude that there can only exist divisible design graphs on 10, 18, 20, 26, 34, 50 vertices with $\lambda_1 = k - 1$ and a divisible design of class (ii). In Table A.1 all divisible design graphs that are formed from the graphs of Table 5.6 are reported including eigenvalues and multiplicities. Because some parameter sets of the strongly regular graphs correspond to multiple graphs not all divisible design graphs (and strongly regular graphs) are uniquely determined by their spectrum.
The eigenvalues of the divisible design graphs in class (ii) can be found by summing the eigenvalues of the kronecker product $(A \otimes J_2)$ and B, as defined in (5.1). The eigenvalues of the product $A \otimes J_2$ are all possible multiplications of the eigenvalues of A and J_2 and the eigenvalues of B are known. This results in the eigenvalues in Table 5.7, where k, is the valency of the strongly regular graph, v the number of vertices of the strongly regular graph and $r(> 0)$ and $g(< 0)$ are its eigenvalues with corresponding multiplicities m_r and m_g.

Eigenvalues of $A \otimes J + B$	Multiplicity
$2k + 1 = k_{DDG}$	1
$-1 = \alpha_2$	v
$2r + 1 = \alpha_3$	m_r
$2g + 1 = \alpha_4$	m_g

Table 5.7: Eigenvalues of $A \otimes J + I$

The first eigenvalue is simply the degree of the divisible design graph. Furthermore we see that the eigenvalue -1 has multiplicity v which is exactly equal to $m(n-1)$, since $n = 2$.
The first new DDG from class (iii) we can find on 18 vertices, the search procedure of 5.1 resulted in 9 options with $\lambda_1 = k - 1$. Since there is only 1 strongly regular graph on 9 vertices and the divisible design cannot be in class (iii), this can be reduced to only 4 options with the correct $m, n, k, \lambda_1, \lambda_2$ and only multiplicities need to be checked with those of the strongly regular graph.

Note that the three options that do not correspond to a divisible design graph, are not in Table B.1.
The graphs defined by Theorem 5.2.3 and 5.2.4 are the only possible divisible design graphs with $\lambda_1 = k-1$. Because class (iii) of Haemers (1991) is defined by the first theorem and class (ii) by the second. Note that the complement of the strongly regular graph construction correspond to the situation that $\lambda_1 = k$.

5.3 Four eigenvalues

In van Dam & Spence (1998) almost all possible graphs with four eigenvalues are reported on 30 vertices or less. Although certainly not all divisible design graphs have four eigenvalues, it is a substantial part of the results after the procedure in section 5.1. Note that all trivial divisible design graphs and those with $\lambda_1 = k-1$ have four eigenvalues. From this point on we will not pay any attention to divisible design graphs with $\lambda_1 = k-1$, because these are already classified.

5.3.1 Walkregularity

Graphs with four eigenvalues are always walk-regular (van Dam, 1995), therefore we can impose a new restriction on the remaining possibilities with only four eigenvalues.

$$\theta_r = \frac{1}{v}\left(k^r + f_1\alpha_1^r + f_2\alpha_2^r + g_1\alpha_3^r + g_2\alpha_4^r\right) \geq 0 \text{ and } \theta_r \in \mathbb{N}\ \forall r \text{ and } \theta_r \equiv 0 \mod 2 \text{ if } r \text{ odd} \tag{5.2}$$

This equation is a generalization of the fact that in a walk-regular graph the number of closed walks of length r is constant for all vertices and of course must be integer larger or equal to zero. Using this extra condition for $r=3$ and $r=4$ (it already holds for $r=2$), results in excluding 6 options out of the 36 left with four eigenvalues $v \leq 30$. These 36 options include the two options on 8 vertices, we already investigated. Note that options for which $k^2 - \lambda_2 v = 0$ always have 4 eigenvalues.
Options with four eigenvalues that do not satisfy 5.2 are not in Table B.1.

5.3.2 Non-existing spectra

The list with 30 options can be checked with the spectra reported in van Dam & Spence (1998). In Table 5.8 all the options that do not occur in van Dam & Spence are reported, since the spectra of these sets do not exist, the parameter sets do not correspond to divisible design graphs.

Index	v	k	λ_1	λ_2	m	n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$
1	8	3	0	1	4	2	1.732^2	-1.732^2	-	-1^3
2	12	7	3	4	4	3	2^3	-2^5	-	-1^3
3	16	4	0	1	4	4	2^5	-2^7	0^3	-
4	16	7	0	3	8	2	2.646^4	-2.646^4	-	-1^7
5	16	12	8	9	4	4	2^3	-2^9	0^3	-
6	20	13	9	8	4	5	2^7	-2^9	-	-3^3
7	24	5	0	1	6	4	2.236^9	-2.236^9	-	-1^5
8	24	11	0	5	12	2	3.317^6	-3.317^6	-	-1^{11}
9	24	15	10	9	6	4	2.236^9	-2.236^9	-	-3^5
10	24	16	12	10	4	6	2^3	-2^{17}	4^3	-
11	28	19	15	12	4	7	2^{11}	-2^{13}	-	-5^3

Table 5.8: Parameter sets with non-existing spectra

5.3.3 Uncertain existence & Distance-regularity

There are now 19 options left on 30 vertices or less with four eigenvalues that may correspond to divisible design graphs. Out of these 19 options there are 4 spectra for which the classification is not yet finished, see Table 5.9.

Index	v	k	λ_1	λ_2	m	n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$	Lower bound
1	28	13	0	6	14	2	3.606^7	-3.606^7	-	-1^{13}	515
2	28	10	6	3	7	4	2^{14}	-2^7	-	-4^6	2
3	28	15	6	8	7	4	3^7	-3^{14}	1^6	-	8472
4	30	16	12	8	10	3	2^{15}	-2^5	-	-4^9	82

Table 5.9: Parameter sets with four eigenvalues that are not completely classified

The parameter sets can correspond to divisible design graphs, but since not all graph with these spectra are determined we cannot tell how many divisible design graphs have these parameters. On the web page of Spence (n.d.) the hexadecimal representations of the graphs with four eigenvalues can be found. Since option 2 in Table 5.9 has a lower boundary of 2, it was little effort, to check for these two graphs whether they are divisible design graphs. It turns out that these two graphs are no divisible design graphs. For option 1, 3 and 4 the graphs that are known, are not yet checked. We can however exclude option 2 and 4 completely. The trace of the matrix R belonging to the divisible design graph of option 2 should be equal to $10 - 4 \cdot 6 = -14$, clearly it is not possible to have a negative diagonal sum of R. For option 4 holds the same, the trace of R should equal -20, that is also not possible, see Corollary 4.1.2.

Conjecture 5.3.1 *If a divisible design graph is cospectral with a distance regular graph, it is a distance-regular.*

This conjecture has its origin in the fact that out of the options on four eigenvalues 6 of these also correspond to one of the distance-regular graphs in Haemers & Spence (1995). For option 18 in Haemers & Spence (13 in Table 5.10) it is remarkable that it has exactly 2 distance-regular graphs, whereas there are 13 graphs with that spectrum (van Dam & Spence, 1998) and we have also found exactly 2 divisible design graphs with this spectrum with the files of Spence. Though we have not checked that the two divisible design graphs we have found are indeed $\overline{\mathrm{GQ}(2,4)\backslash\text{spread}}$, i.e. minus 2 different spreads, but we are sure there are 2.
If this conjecture is correct option 1 in Table 5.9 corresponds to exactly 1 divisible design graph, the Taylor graph. In Haemers & Spence (1995) it is shown that there is only one graph with that spectrum that is distance-regular and therefore it may be the only divisible design graph.
If Conjecture 5.3.1 is true, we also know that parameter set 7 in Table 5.10 corresponds to only one divisible design graph, that is J(6,3), because it is the only distance-regular graph. The Johnson graph (6,3) can shown to be a divisible design graph, the other 8 graphs with the spectrum are reported in Haemers & Spence (1995), but not checked.
The final example with this conjecture is the Klein graph, the spectrum of the Klein graph admits 10 non-isomorphic graphs, only one of these is a divisible design graph, and we can show that that is the Klein graph (Option 8 in Table 5.10).

5.3.4 Constructions of graphs

The spectrum of option 9 in Table 5.10 exists, but does not correspond to a divisible design graph. There exist 5 graphs with this spectrum (van Dam & Spence, 1998) all of these cannot be arranged in groups such that λ_1 and λ_2 are the correct constants.
The spectra of option 7, 8, 12, 13 all allow for more than one graph, but there exist 1, 1, 4 and 2 divisible design graphs with these spectra. (Option 7 needs to be checked or conjecture proven, 13 only to be sure of the name of the graph).

Index	v	k	λ_1	λ_2	m	n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$	# graphs	Notes
1	8	4	0	2	4	2	2^1	-2^3	0^3	-	1	$\overline{\text{Cube}}$
2	12	5	0	2	6	2	2.236^3	-2.236^3	-	-1^5	1	Icosahedron
3	12	5	1	2	4	3	2^2	-2^6	1^3	-	1	L(K(3,4))
4	12	6	2	3	3	4	2^3	-2^6	0^2	-	1	L(CP(3))
5	15	4	0	1	5	3	2^5	-2^5	-	-1^4	1	L(Petersen)
6	20	7	3	2	4	5	2^4	-2^{12}	3^3	-	1	L(K(4,5))
7	20	9	0	4	10	2	3^5	-3^5	-	-1^9	≥ 1	J(6,3)
8	24	7	0	2	8	3	2.646^8	-2.646^8	-	-1^7	1	Klein graph
9	24	8	4	2	4	6	2^{11}	-2^9	-	-4^3	0	-
10	24	8	4	2	4	6	2^5	-2^{15}	4^3	-	1	L(K(4,6))
11	24	14	7	8	8	3	2.646^8	-2.646^8	-	-2^7	1	$\overline{\text{Klein}_{1,3}}$
12	24	16	12	10	4	6	2^9	-2^{11}	-	-4^3	4	-
13	27	18	9	12	9	3	3^6	-3^{12}	0^8	-	2	$\overline{GQ(2,4)\backslash\text{spread}}$?
14	28	9	5	2	4	7	2^6	-2^{18}	5^3	-	1	L(K(4,7))
15	30	23	19	16	2	15	2^{10}	-2^{18}	-	-7^1	1	$\overline{\text{G}}$=2GQ(2,2)

Table 5.10: Parameter sets with spectra that exist

Of course it is interesting to know whether graphs exist on 30 vertices, but if some of the graphs exhibit a structure that holds when v grows, we know much more. It is clear that option 3, 6, 10 and 14 have a common form. They are all line graphs of complete bipartite graphs, with one group of 4 and one unequal to 4.

Theorem 5.3.1 *A L(K$(4, n)$) graph is also a $(4, n, n+2, n-2, 2)$ divisible design graph, $n \neq 4, n > 2$.*

Proof. A line graph can directly found by $CC^T - 2I$, where C is the matrix representing all edges, therefore C is of size $(4n \times (4+n))$. This matrix looks like:

$$C = \begin{pmatrix} A_1 & I_n \\ A_2 & I_n \\ A_3 & I_n \\ A_4 & I_n \end{pmatrix}$$

Where A_i is a $(n \times 4)$ matrix with ones in column i and 0 anywhere else. CC^T is of size $(4n \times 4n)$. Then we get the line graph:

$$L = \begin{pmatrix} J_n - I_n & I_n & I_n & I_n \\ I_n & J_n - I_n & I_n & I_n \\ I_n & I_n & J_n - I_n & I_n \\ I_n & I_n & I_n & J_n - I_n \end{pmatrix}$$

It can be clearly seen that the matrix L above corresponds to a divisible design graph with the desired parameters. n cannot be equal to 4, because then $\lambda_1 = \lambda_2$. If $n = 2$, then $\lambda_2 = 0$, which is a trivial case. □

Note that all graphs defined by Theorem 5.3.1 are circulant. The matrix can also be written as $J_4 \otimes I_n + K - 2I_n$, which can be used to find the multiplicities f_1, f_2, g_1, which are $n-1, 3(n-1), 3$ and g_2 is always zero. Since these graphs have eigenvalue -2, there cannot be other graphs with the spectrum of $L(K(4, n))$.

Option 12 in Table 5.10 is found by computer search only , done by Van Dam & Spence (1998), and therefore does not yield any new construction methods.

Option 15 is the only option that has an eigenvalue $k - v$ and therefore the complement of the graph is disconnected and the disjoint union of strongly regular graphs, in case of divisible design graphs this disjoint union consists of strongly regular graphs with a complement that has $\lambda = \mu$. In the list for $v \leq 30$ with four eigenvalues option 15 is the only one with an eigenvalue $k - v$, this construction method may be used for larger v.

Theorem 5.3.2 *If A is the adjacency matrix of a (v, k, λ)-graph and $k \neq v - 1$, then $I_m \otimes A + J_{mv} - K_{mv}$ is the adjacency matrix of a divisible design graph.*

Proof. We have $B = I_m \otimes A + J_{mv} - K_{mv}$. The number of groups is m and the group size is v. The valency is $k + v(m - 1)$. The inproduct within a group is equal to λ and v times $m - 1$. The inproduct between two points outside is obviously equal to $2\lambda + v(m - 2)$. So it is a divisible design and is symmetric and has a zero diagonal, therefore it is a divisible design graph. □

The eigenvalues are $k + mv - v = k + v(m - 1)$ with multiplicity 1 and is the degree of the DDG. The other eigenvalues are $\pm\sqrt{k - \lambda}$ and this is equal to the eigenvalues of the (v, k, λ)-graph, the multiplicities also correspond. α_3 does not exist and α_4 is equal to $k + v(m - 1) - vm = k - v$. All eigenvalues are expressed in the parameters of the $(vk, k\lambda)$-graph.
Since there are only 3 (v, k, λ)-graphs smaller than 25 (Brouwer n.d.), there are only 6 divisible design graphs of this type on 50 vertices or less. These are the ones with index 100, 115, 122, 284, 385, 398 in Table B.1. Since graphs with eigenvalue $k - v$ are always such that the complement is a disjoint union of strongly regular graphs, these are the only possibilities the spectrum of the divisible design graphs admits.

Option 4 and 5 (and the ones defined by Theorem 5.3.1) are graphs with least eigenvalue -2. Which means the graph G can only be (van Dam & Spence (1998):

i. C_7

ii. The line graph of a strongly regular graph

iii. The line graph of the incidence graph of a square? design

iv. The line graph of a complete bipartite graph

v. One of $BCS_9, BCS_{70}, BCS_{153} - BCS_{160}, BCS_{179}, BCS_{183}$ found by Bussemaker, Cvetković and Seidel (1978).

It is clear that option 4 and 5 are both line graphs of strongly regular graphs, since both the Petersen and the cocktail party graph are strongly regular. The options defined by Theorem 5.3.1 are line graphs of complete bipartite graphs (iv). Note that all graphs in (v) never correspond to divisible design graphs. It is not true that the line graph of every strongly regular graph is a divisible design graph, take for example the strongly regular graph on 9 vertices, the Paley(9) graph.
The other options (1,2,7,8,11,13) are all constructed with association schemes, see van Dam & Spence (1998).
Van Dam (1995) shows that a graph with four eigenvalues and two of these eigenvalues (k and μ) are simple, i.e. with multiplicity one, can be partitioned into two parts of equal size, such that every vertex has $\frac{1}{2}(k + \mu)$ neighbors in its own part and $\frac{1}{2}(k - \mu)$ in the other part. Although on 30 vertices or less this result does not exclude possibilities, it might be useful for larger v.

A final not so obvious construction we have already seen with the line graph of the Petersen graph. It can also be written as: L(Petersen)=$J_{15} - K_{53} - A$, where A is the adjacency matrix of the triangular graph T(6). Of course this is not true for every A.
It must be true that $(J - K - A)^2$ equals (3.4). We can easily see that if A is a k-regular graph, with a partition into classes of size n, then $JA = AJ = kJ$, $JK = KJ = nJ$, $J^2 = vJ$. The

partition of A must have zero blocks on the diagonal to avoid negative entries, therefore it has a Hoffman coloring.
If the row sum per block of A is constant $AK = KA = Q \otimes J_n$, and Q is the matrix with the row sums. In order to satisfy (3.4) A can only correspond to a (v, k, λ)-graph and the row sums of the off-diagonal blocks of A must be a constant, γ, thus $AK = KA = \gamma(J - \gamma)K$. Therefore:

$$(J - A - K)^2 = (v - 2k - 2n + \lambda + 2\gamma)J_v + (k - \lambda)I_v + (n + -2\gamma)K \tag{5.3}$$

So we can express the paramaters of the divisible design in the parameters of the (v, k, λ)-graph:

$$\lambda_2 = v - 2k - 2n + \lambda + 2\gamma \tag{5.4}$$
$$\lambda_1 = v - 2k - n + \lambda \tag{5.5}$$
$$k_{DDG} = v - k - n \tag{5.6}$$

This leads to the following theorem:

Theorem 5.3.3 *If there exists a (v, k, λ)-graph with an equitable partition with zero diagonal blocks ($n \times n$) such that the row sums γ of the off-diagonal blocks are equal ($n \neq 2\gamma$) and with a Hoffman coloring, then $J_v - A - K$ is a divisible design graph.*

In case of the line graph of the Petersen graph, we already saw that $A =$T(6).
There is also a (v, k, λ)-graph on 16 vertices, but it does not appear in Table B.1. So it does not have the desired equitable partition. The next (v, k, λ)-graph is (35,18,9), this graph has a Hoffman coloring. This would correspond to a $(7, 5, 12, 3, 4)$ divisible design graph, in Table B.1 there are two options (130, 131) with these parameters, so we need to check which of the multiplicities is correct.
We need the eigenvalues of $J - A - K$, these eigenvalues are in Table 5.11 expressed in the parameters of the (v, k, λ)-graph. The first eigenvalue α_0 can be found by summing all eigenvalues corresponding to 1_v.
Since we know that $J - A - K$ has a Hoffman coloring and a constant row sum in the off-diagonal blocks, it has $R = (n - \gamma)(J_m - I_m)$. Therefore R has the eigenvalue $(n - \gamma)(m - 1)$ with multiplicity 1 and $-(n - \gamma)$ with multiplicity $m - 1$. We know that this last value is either α_3 or α_4, since it is negative it must be α_4. The other eigenvalues are $\pm\sqrt{k_{DDG} - \lambda_1} = \pm\sqrt{(v - k - n) - (v - 2k - n + \lambda)} = \pm\sqrt{k - \lambda}$. The multiplicities can be computed from the trace, that must be zero. Since $\alpha_0 + (m - 1)\alpha_4 = 0$, the multiplicities of α_1 and α_2 must be the same.

Eigenvalues of $J - A - K$ (5.3.3)	Multiplicity
$\alpha_0 = v - k - n$	1
$\alpha_1 = \sqrt{k - \lambda}$	$\frac{1}{2}(m(n - 1))$
$\alpha_2 = -\sqrt{k - \lambda}$	$\frac{1}{2}(m(n - 1))$
$\alpha_4 = -(n - \gamma) = -\frac{v-k-n}{m-1}$	$m - 1$

Table 5.11: Eigenvalues of $J - A - K$ expressed in parameters of(v, k, λ)-graph A

The first divisible design graph that is defined by Theorem 5.3.3 is the one with index 130 in Table B.1. There might also be such a divisible design graphs on 36, 40 and 45 vertices, we have not yet checked.

5.4 Five eigenvalues

After the search procedure in section 5.1 65 options with five eigenvalues are left on 30 vertices or less.

5.4.1 $\lambda_1 = 0$

The case that $\lambda_2 = 0$ has already been thoroughly explained and can be regarded as a trivial design. The case $\lambda_1 = 0$ is not trivial, but has characteristics that makes it worthwhile to investigate all remaining options with $\lambda_1 = 0$.
In Bose (1977) it is proven that square divisible designs with $\lambda_1 = 0$ have an incidence matrix N that is built of blocks of $(n \times n)$ and these blocks have a constant row and column sum. In the case of $\lambda_1 = 0$ this sum is 0 or 1. Furthermore he shows that if a square divisible design with $\lambda_1 = 0$ exists, there also exists an square 2-design D_0 with parameters m, k_0, λ_0, where $k_0 = m - k$ and $\lambda_0 = m - 2k + n\lambda_2$.

Corollary 5.4.1 *A divisible design graph with $\lambda_1 = 0$ has $m \geq k$ and $m - 2k + n\lambda_2 \geq 0$.*

Proof. Suppose $m < k$. We know from Bose (1977) that if $\lambda_1 = 0$ the column and row sums of the blocks are 0 or 1. This means every row has a maximum of m non-zero entries. We assumed that $m < k$, which results in a contradiction and therefore $m \geq k$.
From Bose we also know that if $\lambda_1 = 0$ and the divisible design has the dual property, there must exist a 2-design with $\lambda_0 = m - 2k + n\lambda_2$. A divisible design graph possesses the dual property, thus must fulfill this condition, which means λ_0 must be greater or equal to 0. □

Although Corollary 5.4.1 does not exclude any of the options in Table 5.12 it might be useful for larger v or for different search procedures.
In Table 5.12 all parameter sets with $\lambda_1 = 0$ and five eigenvalues are presented for $v \leq 30$.

Index	v	k	λ_1	λ_2	m	n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$
1	15	4	0	1	5	3	2^4	-2^6	1^2	-1^2
2	20	9	0	4	10	2	3^4	-3^6	1^3	-1^6
3	20	9	0	4	10	2	3^3	-3^7	1^6	-1^3
4	26	9	0	3	13	2	3^5	-3^8	1.732^6	-1.732^6

Table 5.12: Possible parameter sets with $\lambda_1 = 0$ and $v \leq 30$

Apart from m, k_0 and λ_0 that should form a design, the matrix R from Bose(1977) should in itself also correspond to a 2-design, with parameters $(m, k, n\lambda_2)$.
This means the matrix R in case of option 1 should be a 2-(5,4,3) design. This design does exist. For a divisible design graph this matrix R must also be symmetric. The only numbers in this matrix R are zeros and ones, one zero in every row (and column). If R has a constant diagonal of zeros or ones, the divisible design graph is walk regular (Theorem 4.1.2) and must satisfy condition (5.2). Furthermore the diagonal sum must equal $k + (g_1 - g_2)\alpha_3$ (Corollary 4.1.2) and the row sum of the matrix must be equal to k. Thus in the case of option 1 there must be 4 ones on the diagonal. This is not possible because there is no symmetric (3×3) matrix with row sum 1 and a zero diagonal: option 1 does not correspond to a divisible design graph.
Option 3 is also no divisible design graph, the row sum of R should be equal to 12 (Corollary 4.1.2), which is impossible, because the sum of any block is equal to 0 or 1.
The diagonal sum of R is equal to 6 for option 2 and the row sum to 9. For option 4 both sums must be equal to 9.

5.4.2 Bose

Bose's result (1977) in the previous paragraph is actually a generalization of another theorem. Which states that if there exists a $t \in \mathbb{R}, t \neq 1$ that satisfies:

$$2k = mt + \frac{(n-1)\lambda_1}{t-1} \tag{5.7}$$

then the matrix R has only values t or $t-1$ and there exists a 2-(m, k_0, λ_0) design, where:

$$k_0 = k - \frac{(n-1)\lambda_1}{t-1}, \qquad \lambda_0 = n\lambda_2 - \frac{t(n-1)\lambda_1}{t-1} \tag{5.8}$$

Out of the 65 options with five eigenvalues, there are 9 parameter sets that admit a t, see Table 5.13.

Index	v	k	λ_1	λ_2	m	n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$	t
1	12	5	1	2	4	3	2^3	-2^5	1^1	-1^2	2
2	12	7	3	4	4	3	2^2	-2^6	1^2	-1^1	2
3	14	10	6	7	7	2	2^1	-2^6	1.414^3	-1.414^3	2
4	20	11	2	6	10	2	3^4	-3^6	1^2	-1^7	2
5	20	11	2	6	10	2	3^3	-3^7	1^5	-1^4	2
6	20	11	2	6	10	2	3^2	-3^8	1^8	-1^1	2
7	28	13	4	6	7	4	3^9	-3^{12}	1^1	-1^5	2
8	28	13	4	6	7	4	3^8	-3^{13}	1^4	-1^2	2
9	28	15	6	8	7	4	3^8	-3^{13}	1^3	-1^3	3

Table 5.13: Possible parameter sets with t satisfying (5.7)

Option 1 and 2 in Table 5.13 have their smallest eigenvalue equal to -2. In the previous section we stated some conditions for graphs with four eigenvalues and least eigenvalue -2, in general the following theorem holds.

Theorem 5.4.1 (Cvetković, Rowlinson & Simić, 2004) *If G is a regular connected graph with least eigenvalue -2, then one of the following holds:*

i. G is a line graph

ii. G is a cocktail party graph

iii. G is an exceptional graph (with a representation in E_8).

Option 1 and 2 are no exceptional graphs because of a theorem in Cvetković et al (2004, p. 90). That theorem states that any exceptional graph should fulfill one of three conditions,, option 1 and 2 fulfill neither.
Therefore option 1 and 2 can only be line graphs of some sort. From Cvetković et al. (2004, p. 5) we know that for a connected regular line graph $L(H)$ it must hold that H is regular or H is semi-bipartite. Thus in our case H there are 5 possibilities for H:

a. 2-regular on 12 vertices, cycle graph

b. 3-regular on 8 vertices

c. 4-regular on 6 vertices, cocktail party graph

d. complete bipartite graph K(3,4)

e. complete bipartite graph K(2,6)

We can already exclude option d. as a possibility for H, this is indeed a divisible design graph, and already classified in Theorem 5.3.1, option 3. Obviously option a. is also not one of the desired divisible design graphs. The 3-regular graphs on 8 vertices all result in 4-regular line graphs and we need a 5 or 7 regular graph. The cocktail party graph on 6 vertices, has a 6-regular line graph and also does not correspond to option 1 or 2 in Table 5.13. The complete bipartite $K(2,6)$ also has a 6-regular line graph and therefore we can conclude that option 1 and 2 cannot exist.

Option 3,5,6 from Table 5.13 can be excluded because of Corollary 4.1.2. Option 4 can be excluded because the diagonal sum D of R is $D < (t-1)m$ and option 8 because $D > mt$.
Option 7 and 9 in Table 5.13 might still be possible.

5.4.3 Constructions of graphs

There are now 56 options left, of which 5 are on $v \leq 20$ including option 2 in Table 5.12, reprinted as option 4 in Table 5.14. We will try to reconstruct these 5 graphs or show they cannot exist.

Index	v	k	λ_1	λ_2	m	n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$
1	18	9	6	4	6	3	1.732^6	-1.732^6	3^1	-3^4
2	18	10	6	5	3	6	2^5	-2^{10}	3.162^1	-3.162^1
3	20	7	3	2	4	5	2^7	-2^9	3^1	-3^2
4	20	9	0	4	10	2	3^4	-3^6	1^3	-1^6
5	20	13	9	8	4	5	2^4	-2^{12}	3^2	-3^1

Table 5.14: Remaining parameter sets for $v \leq 20$

We will start with examining option 1 in Table 5.14 more closely. The diagonal of the matrix R must sum up to zero, since the sums of the blocks are nonnegative, R has a zero diagonal. The row sum of R must be equal to nine, there are a few combinations that sum up to 9 and have at least one zero, see Table 5.15.

Index	$\# r_{ij} = 0$	$\# r_{ij} = 1$	$\# r_{ij} = 2$	$\# r_{ij} = 3$	Inproduct
1	3	0	0	0	9
2	2	1	1	2	7
3	2	0	3	1	6
4	1	3	0	2	6
5	1	2	2	1	5

Table 5.15: Possible entries for R for option 1 (Table 5.14)

The inproduct of a block of zeros is obviously zero. The inproduct of a matrix with row and column sum equal to one is also zero. The inproduct of J_n is n, in this case 3 and because we have (3×3) matrices, the inproduct with row sum 2 is 1. Therefore it can easily be seen that only option 3 and 4 might be entries for R, because the inproduct equals λ_1.
Option 3 in Table 5.15 cannot be the way to construct a divisible design graph, because it is not possible the place the blocks such that the inproduct outside the groups is equal to λ_2, 4. This can be easily seen, when you try any arbitrary combination of the first row of R. If the second zero block (the first is on the diagonal) is in column i then it is not possible to place the three blocks with sum two and the one with three, such that $\lambda_2 = 4$.
Option 4 however can correspond to a divisible design graph with the desired parameters. One possibility for an adjacency matrix is:

$$A = \begin{pmatrix} 0_3 & I_3 & I_3 & I_3 & J_3 & J_3 \\ I_3 & 0_3 & J_3 & J_3 & I_3 & I_3 \\ I_3 & J_3 & 0_3 & J_3 & b_3 & a_3 \\ I_3 & J_3 & J_3 & 0_3 & a_3 & b_3 \\ J_3 & I_3 & a_3 & b_3 & 0_3 & J_3 \\ J_3 & I_3 & b_3 & a_3 & J_3 & 0_3 \end{pmatrix}, \text{ where } a_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } b_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Therefore we know that there exists at least one divisible design graph with the parameters of option 1 in Table 5.14.
For option 2 in Table 5.14 we can try to construct a matrix R with row sum equal to 10, diagonal sum equal to 10 (sum of eigenvalues) and entries from zero to 6. There are only few such possibilities (7) for three digits to sum up to 10, some of these will not correspond to the desired R, because λ_1 will not be equal to 6. The eigenvalues of the remaining (3×3) matrices however do not correspond to 10, $\sqrt{10}$ and $-\sqrt{10}$. Option 2 is no divisible design graph.

Option 3 must have an R with diagonal sum of 4 and a row sum of 7. Since all blocks are of size (5×5), there cannot be a diagonal block with sum 1, because there is no symmetric matrix with row sum 1 and zeros on the diagonal. Therefore the diagonal of R is (4,0,0,0) or (2,2,0,0) both in all possible orders. If the diagonal is (4,0,0,0), R can be one of the following matrices (rows maybe interchanged if the corresponding columns are also interchanged):

$$R_1 = \begin{pmatrix} 4 & 3 & 0 & 0 \\ 3 & 0 & 2 & 2 \\ 0 & 2 & 0 & 5 \\ 0 & 2 & 5 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 4 & 2 & 1 & 0 \\ 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 4 \\ 0 & 3 & 4 & 0 \end{pmatrix}, R_3 = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 0 & 3 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & 3 & 3 & 0 \end{pmatrix}$$

Computing the eigenvalues of R_1, R_2, R_3 shows that R_3 has the desired eigenvalues 7,3,-3,-3. If the diagonal is (2,2,0,0) there cannot be found a matrix R that has the correct eigenvalues.
If R_3 is indeed an R that corresponds with a divisible design graph, the blocks on the diagonal are $J_5 - I_5$ and zero blocks, on the first row and column there are furthermore identity matrices (any other permutation matrix is also fine). For the other rows of blocks to have an inproduct $\lambda_2 = 2$ with the first block row, the following must hold:

$$A = \begin{pmatrix} J_5 - I_5 & I & I & I \\ I & 0_5 & B & C \\ I & B^T & 0_5 & C^T \\ I & C^T & B^T & 0_5 \end{pmatrix} \text{where}\, B + C = J_5 + I_5 \,\text{and}\, diag(B) = diag(C) = (1,1,1,1,1)$$

Know look at the inproduct of block row 2 and 3. For the sixth row of A to have an inproduct of $\lambda_2 = 2$ with the rows of the third block row (row 11 to 15 of A), we need to have again $C + C^T = J + I$. This implies that $B = C^T$, thus we have:

$$A = \begin{pmatrix} J_5 - I_5 & I & I & I \\ I & 0_5 & B & B^T \\ I & B^T & 0_5 & B \\ I & B & B^T & 0_5 \end{pmatrix}$$

This matrix B cannot be symmetric because then the inproduct within a group, $\lambda_1 = 3$, is not correct. It would imply that within the block all rows have an inproduct of 1.5. We need to try to construct a matrix B such that $B + B^T = J + I$ such that every row of the matrix $[B\, B^T]$ has 3 ones in common with any other row. Let's start the matrix B with an arbitrary row (1,1,1,0,0), then we know the following:

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & a & b & c \\ 1 & * & 1 & * & * \\ 1 & * & * & 1 & * \\ 1 & * & * & * & 1 \end{pmatrix} \quad B^T = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & d & e & f \\ 1 & * & 1 & * & * \\ 0 & * & * & 1 & * \\ 0 & * & * & * & 1 \end{pmatrix}$$

Row 1 and 2 of the matrix $[B\, B^T]$ now have 2 ones in common, we need only a one more. Suppose $a = 1$ and b or c is zero, then either e or f is one and then there are too much ones in common. If $a = 0$ then $b = c = 1$ and $e = f = 0$ and we have too little ones in common.
Whatever row you start with, when constructing B, this problem always arises, therefore option 3 in Table 5.14 does not exist.
Option 4 in Table 5.14 can only have a matrix R that is as follows:

$$R = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The block size is in this case only two, therefore we have three matrices that can be used to build the divisible design graph:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J - I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We can now represent the adjacency matrix A with another (10×10) matrix, X where 0 represents 0_2, 1 represents the matrix I_2 and -1 represents $J_2 - I_2$. Obviously there are 10 zeros in this matrix and the matrix is symmetric. The inproduct outside the groups must equal 4, every row contains exactly one zero, therefore precisely four entries are identical and in the same position as in any other row. For example row 1 of X is (1,0,1,1,1,1,-1,-1,-1,-1), then a correct row 2 would be (0,1,1,1,-1,-1,-1,-1,1,1) of course this must also hold for row 3 to 10 and between any other pair of rows. the inproduct of two different rows is 0 now, thus:

$$XX = 9I \text{ and } XX^{-1} = I \longrightarrow X^{-1} = \frac{1}{9}X \tag{5.9}$$

This means the matrix X corresponding to R, is almost an orthogonal matrix. The matrix X looks as follows[3]:

$$X = \begin{pmatrix} -1 & 0 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 0 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 0 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 0 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 0 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & -1 & 0 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 0 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}$$

The adjacency matrix is the found by substituting for I_2, $J_2 - I_2$ and 0_2. Note that this graph is the first divisible design graph that is not walk-regular.
The only possibility for option 5 in Table 5.14 for the diagonal of R is (4,4,4,4), since it must sum up to 16. To get a row sum of 13 we need to check some possibilities of (4×4) matrices, if the row sum of R must be 13, the only R that gives the correct eigenvalues, is the one where the row is built off (4,4,4,1). It can be easily seen that the matrix below indeed corresponds to a divisible design graph with the parameters of option 5 in Table 5.14.

$$A = \begin{pmatrix} J_5 - I_5 & J_5 - I_5 & J_5 - I_5 & I_5 \\ J_5 - I_5 & J_5 - I_5 & I_5 & J_5 - I_5 \\ J_5 - I_5 & I_5 & J_5 - I_5 & J_5 - I_5 \\ I_5 & J_5 - I_5 & J_5 - I_5 & J_5 - I_5 \end{pmatrix}$$

[3]With help from Willem Haemers who put the 1's and -1's at the right positions

This adjacency matrix A exhibits a clear pattern:

$$A = (J_m - X_m) \otimes (J_n - I_n) + X_n \otimes I_n \quad \text{where} \quad X_m = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \vdots & \iddots & & \vdots \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{m \times m} \tag{5.10}$$

If this construction exists the matrix R is defined as:

$$R = (n-1)J_m + (2-n)X_m \tag{5.11}$$

The eigenvalues are $k = mn - m - n + 2$ and $\pm(2-n)$. We know from Corollary 4.1.3 that the eigenvalues and multiplicities of A are partly equal to those of R, thus $(n-2) = \sqrt{k^2 - \lambda_2 mn}$. Since we can express k in terms of m, n, we can rewrite $\lambda_2 = mn - 2m - n + 6 + \frac{m-4}{n}$. This results in a necessary condition.

Corollary 5.4.2 *If $A = (J_m - X_m) \otimes (J_n - I_n) + X_n \otimes I_n$ is the adjacency matrix of a divisible design graph, we have $m - 4 \equiv 0 \mod n$.*

Proof. If $m - 4 \not\equiv 0 \mod n$ then λ_2 is not an integer. □

We now know k, λ_2, m, n, therefore with (3.6) we could express λ_1 in m, n and possibly find some new divisibility conditions.

Chapter 6

Concluding remarks

At this moment all spectra of divisible design graphs on 20 vertices or less are known. For just four instances it is uncertain how many (non-isomorphic) divisible design graphs correspond to a certain spectrum. Although not all of the graphs are reported in this thesis, they can all be found with the theorems, since for small v all combinatorial objects needed are known.
On 50 vertices and less we have found 35 parameter sets for which existence is confirmed and there are 241 left for which existence is unclear.
At the end of this thesis a critical comment might be that the trivial cases of Chapter 4 are not at all trivial. The existence of 2-designs or (v, k, λ)-graphs is not trivial at all. They are only trivial in the sense that we know how to construct these cases, although we do not (yet) have the building material. The class of graphs defined by Theorem 5.2.3 is much more trivial.

An obvious open end in this thesis is quite obviously the proof or counterexample for Conjecture 5.3.1, proof would exclude some uncertainties in Table B.1.
The computer search used in this thesis is obviously not efficient. It includes spectra that van Dam & Spence were able to exclude earlier. Furthermore the search procedure described in 5.1 does not include all knowledge and therefore is not as efficient as it could have been. For $v \leq 50$ it takes only a few seconds to find possible parameter sets, for large v the procedure needs updating.

Appendix A

DDG's with $\lambda_1 = k - 1$

In Section 5.2 all possible DDG's for $\lambda_1 = k - 1$ have been defined. Haemers (1991) classified all GDD with $\lambda_1 = k - 1$ and with this in mind Theorem 5.2.3 summarizes all possible DDG's of class (iii).
Theorem 5.2.4 summarizes all DDG's of class (ii). Because the existence of strongly regular graphs is not in all cases evident or trivial, these DDG's are present in a separate table.

Index	v	k	λ_1	λ_2	m	n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$	# graphs
1	10	5	4	2	5	2	-	-1^5	$\sqrt{5}^2$	$-\sqrt{5}^2$	1
2	18	9	8	4	9	2	-	-1^9	3^4	-3^4	1
3	20	7	6	2	10	2	-	-1^{10}	3^5	-3^4	1
4	20	13	12	8	10	2	-	-1^{10}	3^4	-3^5	1
5	26	13	12	6	13	2	-	-1^{13}	$\sqrt{13}^6$	$-\sqrt{13}^6$	1
6	34	17	16	8	17	2	-	-1^{17}	$\sqrt{17}^8$	$-\sqrt{17}^8$	1
7	50	25	24	12	25	2	-	-1^{25}	5^{12}	-5^{12}	15

Table A.1: DDG's with $\lambda_1 = k - 1$ and in class (ii)

Note that for example option 4 is built from the complement of the strongly regular graph used for opti0n 3.

Appendix B

List of possible parameter sets

In Table B.1 all possible parameter sets are reported on 50 vertices or less. The parameters are found by the search procedure of section 5.1. This means this table excludes the cases $\lambda_1 = k$ and $\lambda_2 = 0$. Furthermore the divisible design graphs defined by Theorem 5.2.3 are not included in the list.

Since it is known that graphs with four eigenvalues are walk-regular, all possibilities with four eigenvalues are checked whether they satisfy the condition for walk-regularity, (5.2) for $r = 3$ and $r = 4$. Parameter sets that did not satisfy (5.2) are excluded from the table.
In Table B.1 all parameters are given, together with the eigenvalues α_i and multiplicities f_i and g_i. If there exists a t such that (5.7) is satisfied, this or these t's are reported.
The number of graphs, denoted by # has a zero if the parameter set does not correspond with a divisible design graph, if it is blank existence is not yet know, otherwise it gives the number of (non-isomorphic) graphs or the lower boundary on the number of graphs.
The column 'Notes' gives the name of the graph, if it has one, or the reason why it does not exist if this can be easily stated. The short hand vD&S refers to the publication of van Dam & Spence (1998). Sometimes a (not necessarily unique) matrix is given to construct a divisible design graph with.
Non existence can often be shown with the diagonal sum, D, of the matrix R, which cannot be smaller than 0 or larger than $m(n-1) = D_{max}$. The reference column shows where more information can be found, a section in the thesis or a theorem.
The table increases with v, then k and then λ_1.
There are 241 open cases on $v \leq 50$ and 35 for which existence is confirmed, which leaves 133 parameter sets that do not correspond with a divisible design graph.

Table B.1: (Possible) Divisible Design Graphs

Index	v	k	λ_1	λ_2	m	n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$	t	#	Notes	Ref.
1	8	3	0	1	4	2	1.732^2	-1.732^2	-	-1^3	1	0	vD&S	5.3.2
2	8	4	0	2	4	2	2^1	-2^3	0^3	-	1,2	1	$\overline{\text{Cube}}$	5.3.4
3	10	5	4	2	5	2	-	-1^5	2.236^2	-2.236^2		1	C_{10}(1,4,5)	Thm. 5.2.4
4	12	5	0	2	6	2	2.236^3	-2.236^3	-	-1^5	1	1	Icosahedron	5.3.4
5	12	5	1	2	4	3	2^2	-2^6	1^3	-	2	1	L(K(3,4))	5.3.4
6	12	5	1	2	4	3	2^3	-2^5	1^1	-1^2	2	0	$\alpha_{min} = -2$	5.4.2
7	12	6	2	3	3	4	2^3	-2^6	0^2	-	2,3	1	L(CP(3))	5.3.4
8	12	7	3	4	4	3	2^3	-2^5	-	-1^3	2	0	vD&S	5.3.2
9	12	7	3	4	4	3	2^2	-2^6	1^2	-1^1	2	0	$\alpha_{min} = -2$	5.4.2
10	14	10	6	7	7	2	2^1	-2^6	1.414^3	-1.414^3	2	0	$D > D_{max}$	Cor. 4.1.2
11	15	4	0	1	5	3	2^5	-2^5	-	-1^4	1	1	L(Petersen)	Thm. 5.3.3
12	15	4	0	1	5	3	2^4	-2^6	1^2	-1^2	1	0		5.4.1
13	16	4	0	1	4	4	2^5	-2^7	0^3	-	1,2	0	vD&S	5.3.2
14	16	7	0	3	8	2	2.646^4	-2.646^4	-	-1^7	1	0	vD&S	5.3.2
15	16	12	8	9	4	4	2^3	-2^9	0^3	-	3,4	0	vD&S	5.3.2
16	18	9	6	4	6	3	1.732^6	-1.732^6	3^1	-3^4		≥ 1		5.4.3
17	18	9	8	4	9	2	-	-1^9	3^4	-3^4		1		Thm. 5.2.4
18	18	10	6	5	3	6	2^5	-2^{10}	3.162^1	-3.162^1		0		5.4.3
19	20	7	3	2	4	5	2^4	-2^{12}	3^3	-		1	L(K(4,5))	5.3.4
20	20	7	3	2	4	5	2^7	-2^9	3^1	-3^2		0		5.4.3
21	20	7	6	2	10	2	-	-1^{10}	3^5	-3^4		1		Thm. 5.2.4

Table B.1: continued

Index	v	k , λ_1 , λ_2 , m , n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$	t	#	Notes	Ref.
22	20	9 , 0 , 4 , 10 , 2	3^5	-3^5	-	-1^9	1	≥ 1	J(6,3)	5.3.4
23	20	9 0 4 10 2	3^4	-3^6	1^3	-1^6	1	≥ 1		5.4.3
24	20	9 , 0 , 4 , 10 , 2	3^3	-3^7	1^6	-1^3	1	0	$D > D_{max}$	Cor. 4.1.2
25	20	11 2 6 10 2	3^4	-3^6	1^2	-1^7	2	0	$D < (t-1)m$	5.4.2
26	20	11 , 2 , 6 , 10 , 2	3^3	-3^7	1^5	-1^4	2	0	$D > D_{max}$	Cor. 4.1.2
27	20	11 2 6 10 2	3^2	-3^8	1^8	-1^1	2	0	$D > D_{max}$	Cor. 4.1.2
28	20	13 , 9 , 8 , 4 , 5	2^7	-2^9	-	-3^3		0	vD&S	5.3.2
29	20	13 9 8 4 5	2^4	-2^{12}	3^2	-3^1		≥ 1		5.4.3
30	20	13 , 12 , 8 , 10 , 2	-	-1^{10}	3^4	-3^5		1		Thm. 5.2.4
31	21	12 8 6 3 7	2^6	-2^{12}	4.243^1	-4.243^1				
32	24	5 , 0 , 1 , 6 , 4	2.236^9	-2.236^9	-	-1^5	1	0	vD&S	5.3.2
33	24	6 2 1 3 8	2^9	-2^{12}	3.464^1	-3.464^1				
34	24	7 , 0 , 2 , 8 , 3	2.646^8	-2.646^8	-	-1^7	1	1	Klein graph	5.3.4
35	24	8 4 2 4 6	2^{11}	-2^9	-	-4^3		0	$D < 0$	Cor. 4.1.2
36	24	8 , 4 , 2 , 4 , 6	2^5	-2^{15}	4^3	-		1	L(K(4,6))	5.3.4
37	24	8 4 2 4 6	2^9	-2^{11}	4^1	-4^2				
38	24	8 , 4 , 2 , 4 , 6	2^7	-2^{13}	4^2	-4^1				
39	24	9 4 3 6 4	2.236^9	-2.236^9	3^1	-3^4				
40	24	9 , 6 , 3 , 12 , 2	1.732^6	-1.732^6	3^4	-3^7				
41	24	10 2 4 12 2	2.828^6	-2.828^6	2^3	-2^8				
42	24	10 , 3 , 4 , 8 , 3	2.646^8	-2.646^8	2^1	-2^6				
43	24	10 6 3 3 8	2^8	-2^{13}	5.292^1	-5.292^1				
44	24	11 , 0 , 5 , 12 , 2	3.317^6	-3.317^6	-	-1^{11}	1	0	vD&S	5.3.2
45	24	14 6 8 12 2	2.828^6	-2.828^6	2^2	-2^9				
46	24	14 , 7 , 8 , 8 , 3	2.646^8	-2.646^8	-	-2^7		1	$\overline{\text{Klein}_{1,3}}$	5.3.4
47	24	14 10 7 3 8	2^7	-2^{14}	5.292^1	-5.292^1				
48	24	15 , 10 , 9 , 6 , 4	2.236^9	-2.236^9	-	-3^5		0	vD&S	5.3.2
49	24	15 12 9 12 2	1.732^6	-1.732^6	3^3	-3^8				
50	24	16 , 12 , 10 , 4 , 6	2^9	-2^{11}	-	-4^3		4		5.3.4
51	24	16 12 10 4 6	2^3	-2^{17}	4^3	-		0	vD&S	5.3.2
52	24	16 , 12 , 10 , 4 , 6	2^7	-2^{13}	4^1	-4^2				
53	24	16 12 10 4 6	2^5	-2^{15}	4^2	-4^1				
54	24	18 , 14 , 13 , 3 , 8	2^6	-2^{15}	3.464^1	-3.464^1				
55	25	8 4 2 5 5	2^8	-2^{12}	3.742^2	-3.742^2				
56	25	12 , 8 , 5 , 5 , 5	2^7	-2^{13}	4.359^2	-4.359^2				
57	26	9 0 3 13 2	3^5	-3^8	1.732^6	-1.732^6	1			5.4.1
58	26	13 , 12 , 6 , 13 , 2	-	-1^{13}	3.606^6	-3.606^6		1		Thm. 5.2.4
59	27	6 3 1 9 3	1.732^9	-1.732^9	3^3	-3^5				
60	27	8 , 4 , 2 , 9 , 3	2^7	-2^{11}	3.162^4	-3.162^4				
61	27	12 6 5 9 3	2.449^9	-2.449^9	3^2	-3^6				
62	27	16 , 12 , 8 , 3 , 9	2^8	-2^{16}	6.325^1	-6.325^1				
63	27	16 12 9 9 3	2^5	-2^{13}	3.606^4	-3.606^4				
64	27	18 , 9 , 12 , 9 , 3	3^6	-3^{12}	0^8	-	2,3	2	$\overline{\mathrm{GQ}(2,4)\backslash\text{spread}}$?	5.3.4
65	27	20 16 14 3 9	2^7	-2^{17}	4.69^1	-4.69^1				
66	28	6 , 2 , 1 , 7 , 4	2^9	-2^{12}	2.828^3	-2.828^3				
67	28	9 5 2 4 7	2^6	-2^{18}	5^3	-		1	L(K(4,7))	5.3.4
68	28	9 , 5 , 2 , 4 , 7	2^{11}	-2^{13}	5^1	-5^2				
69	28	10 6 3 7 4	2^{14}	-2^7	-	-4^6		0	$D < 0$	Cor. 4.1.2
70	28	10 , 6 , 3 , 7 , 4	2^{12}	-2^9	4^1	-4^5		0	$D < 0$	Cor. 4.1.2
71	28	10 6 3 7 4	2^{10}	-2^{11}	4^2	-4^4				
72	28	10 , 6 , 3 , 7 , 4	2^8	-2^{13}	4^3	-4^3				
73	28	10 6 3 7 4	2^6	-2^{15}	4^4	-4^2				
74	28	10 , 6 , 3 , 7 , 4	2^4	-2^{17}	4^5	-4^1		0	$D > D_{max}$	Cor. 4.1.2
75	28	13 0 6 14 2	3.606^7	-3.606^7	-	-1^{13}	1		Taylor graph?	5.3.3
76	28	13 , 4 , 6 , 7 , 4	3^9	-3^{12}	1^1	-1^5	2			
77	28	13 4 6 7 4	3^8	-3^{13}	1^4	-1^2	2	0	$D > mt$	5.4.2
78	28	15 , 6 , 8 , 7 , 4	3^7	-3^{14}	1^6	-	3			5.3.3
79	28	15 6 8 7 4	3^8	-3^{13}	1^3	-1^3	3			
80	28	18 , 14 , 11 , 7 , 4	2^{10}	-2^{11}	4^1	-4^5				
81	28	18 14 11 7 4	2^8	-2^{13}	4^2	-4^4				
82	28	18 , 14 , 11 , 7 , 4	2^6	-2^{15}	4^3	-4^3				
83	28	18 14 11 7 4	2^4	-2^{17}	4^4	-4^2		0	$D > D_{max}$	Cor. 4.1.2
84	28	18 , 14 , 11 , 7 , 4	2^2	-2^{19}	4^5	-4^1		0	$D > D_{max}$	Cor. 4.1.2
85	28	19 15 12 4 7	2^{11}	-2^{13}	-	-5^3		0	vD&S	5.3.2
86	28	19 , 15 , 12 , 4 , 7	2^6	-2^{18}	5^2	-5^1				
87	28	22 18 17 7 4	2^5	-2^{16}	2.828^3	-2.828^3		0	$D > D_{max}$	Cor. 4.1.2
88	30	6 , 2 , 1 , 15 , 2	2^6	-2^9	2.449^7	-2.449^7				
89	30	10 6 3 15 2	2^5	-2^{10}	3.162^7	-3.162^7				
90	30	16 , 12 , 8 , 10 , 3	2^{15}	-2^5	-	-4^9		0	$D < 0$	Cor. 4.1.2
91	30	16 12 8 10 3	2^{13}	-2^7	4^1	-4^8		0	$D < 0$	Cor. 4.1.2
92	30	16 , 12 , 8 , 10 , 3	2^{11}	-2^9	4^2	-4^7		0	$D < 0$	Cor. 4.1.2
93	30	16 12 8 10 3	2^9	-2^{11}	4^3	-4^6				
94	30	16 , 12 , 8 , 10 , 3	2^7	-2^{13}	4^4	-4^5				
95	30	16 12 8 10 3	2^5	-2^{15}	4^5	-4^4				
96	30	16 , 12 , 8 , 10 , 3	2^3	-2^{17}	4^6	-4^3		0	$D > D_{max}$	Cor. 4.1.2
97	30	16 12 8 10 3	2^1	-2^{19}	4^7	-4^2		0	$D > D_{max}$	Cor. 4.1.2
98	30	18 , 14 , 9 , 3 , 10	2^9	-2^{18}	7.348^1	-7.348^1				
99	30	22 18 15 3 10	2^8	-2^{19}	5.831^1	-5.831^1				
100	30	23 , 19 , 16 , 2 , 15	2^{10}	-2^{18}	-	-7^1		1	A=T(6)	Thm. 5.3.2
101	32	6 0 1 16 2	2.449^8	-2.449^8	2^6	-2^9	1			
102	32	10 , 0 , 3 , 16 , 2	3.162^8	-3.162^8	2^5	-2^{10}	1			
103	32	10 2 3 8 4	2.828^{12}	-2.828^{12}	2^1	-2^6				
104	32	10 , 6 , 2 , 4 , 8	2^{16}	-2^{12}	-	-6^3		0	$D < 0$	Cor. 4.1.2
105	32	10 6 2 4 8	2^7	-2^{21}	6^3	-		1	L(K(4,8))	Thm. 5.3.1
106	32	10 , 6 , 2 , 4 , 8	2^{13}	-2^{15}	6^1	-6^2				
107	32	10 6 2 4 8	2^{10}	-2^{18}	6^2	-6^1				
108	32	14 , 2 , 6 , 16 , 2	3.464^8	-3.464^8	2^4	-2^{11}				
109	32	15 0 7 16 2	3.873^8	-3.873^8	-	-1^{15}	1			

Table B.1: continued

Index	v	k , λ_1 , λ_2 , m , n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$	t	#	Notes	Ref.
110	32	15 , 6 , 7 , 4 , 8	3^{12}	-3^{16}	-	-1^{3}	4			
111	32	16 0 8 16 2	4^{6}	-4^{10}	0^{15}	-	1,2			
112	32	17 , 8 , 9 , 4 , 8	3^{11}	-3^{17}	1^{2}	-1^{1}	5			
113	32	18 6 10 16 2	3.464^{8}	-3.464^{8}	2^{3}	-2^{12}				
114	32	22 , 12 , 15 , 16 , 2	3.162^{8}	-3.162^{8}	2^{2}	-2^{13}	2			
115	32	22 18 12 2 16	2^{12}	-2^{18}	-	-10^{1}		2	A=Shrikhande	Thm. 5.3.2
116	32	22 , 18 , 12 , 2 , 16	2^{7}	-2^{23}	10^{1}	-		0	$D > D_{max}$	Cor. 4.1.2
117	32	22 18 14 4 8	2^{13}	-2^{15}	-	-6^{3}				
118	32	22 , 18 , 14 , 4 , 8	2^{4}	-2^{24}	6^{3}	-		0	$D > D_{max}$	Cor. 4.1.2
119	32	22 18 14 4 8	2^{10}	-2^{18}	6^{1}	-6^{2}				
120	32	22 , 18 , 14 , 4 , 8	2^{7}	-2^{21}	6^{2}	-6^{1}				
121	32	26 20 21 16 2	2.449^{8}	-2.449^{8}	2^{1}	-2^{14}	2			
122	32	26 , 22 , 20 , 2 , 16	2^{10}	-2^{20}	-	-6^{1}		1	A=Clebsch	Thm. 5.3.2
123	32	26 22 20 2 16	2^{7}	-2^{23}	6^{1}	-		0	$D > D_{max}$	
124	33	10 , 0 , 3 , 11 , 3	3.162^{11}	-3.162^{11}	-	-1^{10}	1			
125	33	20 10 12 11 3	3.162^{11}	-3.162^{11}	-	-2^{10}				
126	33	20 , 16 , 10 , 3 , 11	2^{10}	-2^{20}	8.367^{1}	-8.367^{1}				
127	33	24 20 16 3 11	2^{9}	-2^{21}	6.928^{1}	-6.928^{1}				
128	34	17 , 16 , 8 , 17 , 2	-	-1^{17}	4.123^{8}	-4.123^{8}		1		Thm. 5.2.4
129	35	6 0 1 7 5	2.449^{14}	-2.449^{14}	-	-1^{6}	1			
130	35	12 , 3 , 4 , 7 , 5	3^{14}	-3^{14}	-	-2^{6}		3854	A=(35,18,9)	Thm. 5.3.3
131	35	12 3 4 7 5	3^{12}	-3^{16}	2^{3}	-2^{3}				
132	35	12 , 8 , 3 , 5 , 7	2^{12}	-2^{18}	6.245^{2}	-6.245^{2}				
133	35	16 12 6 5 7	2^{11}	-2^{19}	6.782^{2}	-6.782^{2}				
134	35	24 , 18 , 16 , 7 , 5	2.449^{14}	-2.449^{14}	-	-4^{6}				
135	35	28 24 22 7 5	2^{7}	-2^{21}	3.742^{3}	-3.742^{3}				
136	36	9 , 3 , 2 , 12 , 3	2.449^{12}	-2.449^{12}	3^{4}	-3^{7}				
137	36	9 4 2 18 2	2.236^{9}	-2.236^{9}	3^{7}	-3^{10}				
138	36	10 , 6 , 1 , 3 , 12	2^{18}	-2^{15}	-	-8^{2}		0	$D < 0$	Cor. 4.1.2
139	36	10 6 1 3 12	2^{14}	-2^{19}	8^{1}	-8^{1}				
140	36	11 , 7 , 2 , 4 , 9	2^{8}	-2^{24}	7^{3}	-		1	L(K(4,9))	Thm. 5.3.1
141	36	11 7 2 4 9	2^{15}	-2^{17}	7^{1}	-7^{2}				
142	36	12 , 3 , 4 , 4 , 9	3^{14}	-3^{18}	0^{3}	-	3,4			
143	36	14 10 3 3 12	2^{13}	-2^{20}	9.381^{1}	-9.381^{1}				
144	36	17 , 0 , 8 , 18 , 2	4.123^{9}	-4.123^{9}	-	-1^{17}	1			
145	36	22 18 11 3 12	2^{11}	-2^{22}	9.381^{1}	-9.381^{1}				
146	36	24 , 15 , 16 , 4 , 9	3^{12}	-3^{20}	0^{3}	-	6,7			
147	36	25 21 16 4 9	2^{15}	-2^{17}	-	-7^{3}				
148	36	25 , 21 , 16 , 4 , 9	2^{8}	-2^{24}	7^{2}	-7^{1}				
149	36	26 22 17 3 12	2^{6}	-2^{27}	8^{2}	-		0	$D > D_{max}$	Cor. 4.1.2
150	36	26 , 22 , 17 , 3 , 12	2^{10}	-2^{23}	8^{1}	-8^{1}				
151	36	27 21 20 12 3	2.449^{12}	-2.449^{12}	3^{1}	-3^{10}				
152	36	27 , 22 , 20 , 18 , 2	2.236^{9}	-2.236^{9}	3^{4}	-3^{13}				
153	38	9 0 2 19 2	3^{8}	-3^{11}	2.236^{9}	-2.236^{9}	1			
154	38	27 , 18 , 19 , 19 , 2	3^{5}	-3^{14}	2.646^{9}	-2.646^{9}		0	$D > D_{max}$	Cor. 4.1.2
155	39	16 12 6 13 3	2^{9}	-2^{17}	4.69^{6}	-4.69^{6}				
156	39	18 , 9 , 8 , 13 , 3	3^{10}	-3^{16}	3.464^{6}	-3.464^{6}				
157	39	24 20 12 3 13	2^{12}	-2^{24}	10.392^{1}	-10.392^{1}				
158	39	28 , 24 , 18 , 3 , 13	2^{11}	-2^{25}	9.055^{1}	-9.055^{1}				
159	39	30 21 23 13 3	3^{8}	-3^{18}	1.732^{6}	-1.732^{6}	3	0	$D > D_{max}$	Cor. 4.1.2
160	39	32 , 28 , 26 , 13 , 3	2^{5}	-2^{21}	3.162^{6}	-3.162^{6}		0	$D > D_{max}$	Cor. 4.1.2
161	40	9 0 2 10 4	3^{15}	-3^{15}	-	-1^{9}	1			
162	40	9 , 0 , 2 , 10 , 4	3^{14}	-3^{16}	1^{3}	-1^{6}	1			
163	40	9 0 2 10 4	3^{13}	-3^{17}	1^{6}	-1^{3}	1			
164	40	11 , 2 , 3 , 5 , 8	3^{15}	-3^{20}	1^{4}	-	3			
165	40	11 2 3 5 8	3^{16}	-3^{19}	1^{1}	-1^{3}	3			
166	40	12 , 8 , 2 , 4 , 10	2^{21}	-2^{15}	-	-8^{3}		0	$D < 0$	Cor. 4.1.2
167	40	12 8 2 4 10	2^{9}	-2^{27}	8^{3}	-		1	L(K(4,10))	Thm. 5.3.1
168	40	12 , 8 , 2 , 4 , 10	2^{17}	-2^{19}	8^{1}	-8^{2}				
169	40	12 8 2 4 10	2^{13}	-2^{23}	8^{2}	-8^{1}				
170	40	15 , 10 , 5 , 10 , 4	2.236^{15}	-2.236^{15}	5^{3}	-5^{6}				
171	40	16 12 6 20 2	2^{19}	-2^{1}	4^{3}	-4^{16}		0	$D < 0$	Cor. 4.1.2
172	40	16 , 12 , 6 , 20 , 2	2^{17}	-2^{3}	4^{4}	-4^{15}		0	$D < 0$	Cor. 4.1.2
173	40	16 12 6 20 2	2^{15}	-2^{5}	4^{5}	-4^{14}		0	$D < 0$	Cor. 4.1.2
174	40	16 , 12 , 6 , 20 , 2	2^{13}	-2^{7}	4^{6}	-4^{13}		0	$D < 0$	Cor. 4.1.2
175	40	16 12 6 20 2	2^{11}	-2^{9}	4^{7}	-4^{12}		0	$D < 0$	Cor. 4.1.2
176	40	16 , 12 , 6 , 20 , 2	2^{9}	-2^{11}	4^{8}	-4^{11}				
177	40	16 12 6 20 2	2^{7}	-2^{13}	4^{9}	-4^{10}				
178	40	16 , 12 , 6 , 20 , 2	2^{5}	-2^{15}	4^{10}	-4^{9}				
179	40	16 12 6 20 2	2^{3}	-2^{17}	4^{11}	-4^{8}		0	$D > D_{max}$	Cor. 4.1.2
180	40	16 , 12 , 6 , 20 , 2	2^{1}	-2^{19}	4^{12}	-4^{7}		0	$D > D_{max}$	Cor. 4.1.2
181	40	17 8 6 2 20	3^{15}	-3^{23}	7^{1}	-				
182	40	18 , 2 , 8 , 20 , 2	4^{12}	-4^{8}	2^{1}	-2^{18}		0	$D < 0$	Cor. 4.1.2
183	40	18 2 8 20 2	4^{11}	-4^{9}	2^{3}	-2^{16}		0	$D < 0$	Cor. 4.1.2
184	40	18 , 2 , 8 , 20 , 2	4^{10}	-4^{10}	2^{5}	-2^{14}				
185	40	18 2 8 20 2	4^{9}	-4^{11}	2^{7}	-2^{12}				
186	40	18 , 2 , 8 , 20 , 2	4^{8}	-4^{12}	2^{9}	-2^{10}				
187	40	18 2 8 20 2	4^{7}	-4^{13}	2^{11}	-2^{8}		0	$D > D_{max}$	Cor. 4.1.2
188	40	18 , 2 , 8 , 20 , 2	4^{6}	-4^{14}	2^{13}	-2^{6}		0	$D > D_{max}$	Cor. 4.1.2
189	40	18 2 8 20 2	4^{5}	-4^{15}	2^{15}	-2^{4}		0	$D > D_{max}$	Cor. 4.1.2
190	40	18 , 2 , 8 , 20 , 2	4^{4}	-4^{16}	2^{17}	-2^{2}		0	$D > D_{max}$	Cor. 4.1.2
191	40	18 6 8 10 4	3.464^{15}	-3.464^{15}	-	-2^{9}				
192	40	19 , 0 , 9 , 20 , 2	4.359^{10}	-4.359^{10}	-	-1^{19}	1			
193	40	22 6 12 20 2	4^{11}	-4^{9}	2^{2}	-2^{17}		0	$D < 0$	Cor. 4.1.2
194	40	22 , 6 , 12 , 20 , 2	4^{10}	-4^{10}	2^{4}	-2^{15}				
195	40	22 6 12 20 2	4^{9}	-4^{11}	2^{6}	-2^{13}				
196	40	22 , 6 , 12 , 20 , 2	4^{8}	-4^{12}	2^{8}	-2^{11}				
197	40	22 6 12 20 2	4^{7}	-4^{13}	2^{10}	-2^{9}		0	$D > D_{max}$	Cor. 4.1.2
198	40	22 , 6 , 12 , 20 , 2	4^{6}	-4^{14}	2^{12}	-2^{7}		0	$D > D_{max}$	Cor. 4.1.2

Table B.1: continued

Index	v	k , λ_1 , λ_2 , m , n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$	t	#	Notes	Ref.
199	40	22 , 6 , 12 , 20 , 2	4^5	-4^{15}	2^{14}	-2^5		0	$D > D_{max}$	Cor. 4.1.2
200	40	22 , 6 , 12 , 20 , 2	4^4	-4^{16}	2^{16}	-2^3		0	$D > D_{max}$	Cor. 4.1.2
201	40	22 , 6 , 12 , 20 , 2	4^3	-4^{17}	2^{18}	-2^1		0	$D > D_{max}$	Cor. 4.1.2
202	40	23 , 14 , 12 , 2 , 20	3^{14}	-3^{24}	7^1	-				
203	40	24 , 20 , 14 , 20 , 2	2^{19}	-2^1	4^2	-4^{17}		0	$D < 0$	Cor. 4.1.2
204	40	24 , 20 , 14 , 20 , 2	2^{17}	-2^3	4^3	-4^{16}		0	$D < 0$	Cor. 4.1.2
205	40	24 , 20 , 14 , 20 , 2	2^{15}	-2^5	4^4	-4^{15}		0	$D < 0$	Cor. 4.1.2
206	40	24 , 20 , 14 , 20 , 2	2^{13}	-2^7	4^5	-4^{14}		0	$D < 0$	Cor. 4.1.2
207	40	24 , 20 , 14 , 20 , 2	2^{11}	-2^9	4^6	-4^{13}		0	$D < 0$	Cor. 4.1.2
208	40	24 , 20 , 14 , 20 , 2	2^9	-2^{11}	4^7	-4^{12}				
209	40	24 , 20 , 14 , 20 , 2	2^7	-2^{13}	4^8	-4^{11}				
210	40	24 , 20 , 14 , 20 , 2	2^5	-2^{15}	4^9	-4^{10}				
211	40	24 , 20 , 14 , 20 , 2	2^3	-2^{17}	4^{10}	-4^9		0	$D > D_{max}$	Cor. 4.1.2
212	40	24 , 20 , 14 , 20 , 2	2^1	-2^{19}	4^{11}	-4^8		0	$D > D_{max}$	Cor. 4.1.2
213	40	25 , 20 , 15 , 10 , 4	2.236^{15}	-2.236^{15}	5^2	-5^7				
214	40	28 , 24 , 18 , 4 , 10	2^{17}	-2^{19}	-	-8^3				
215	40	28 , 24 , 18 , 4 , 10	2^5	-2^{31}	8^3	-		0	$D > D_{max}$	Cor. 4.1.2
216	40	28 , 24 , 18 , 4 , 10	2^{13}	-2^{23}	8^1	-8^2				
217	40	28 , 24 , 18 , 4 , 10	2^9	-2^{27}	8^2	-8^1				
218	40	29 , 20 , 21 , 5 , 8	3^{13}	-3^{22}	1^1	-1^3	6			
219	40	31 , 22 , 24 , 10 , 4	3^{11}	-3^{19}	1^1	-1^8	4			
220	40	31 , 22 , 24 , 10 , 4	3^{10}	-3^{20}	1^4	-1^5	4			
221	40	31 , 22 , 24 , 10 , 4	3^9	-3^{21}	1^7	-1^2	4	0	$D > D_{max}$	Cor. 4.1.2
222	40	33 , 30 , 27 , 20 , 2	1.732^{10}	-1.732^{10}	3^4	-3^{15}				
223	42	13 , 0 , 4 , 14 , 3	3.606^{14}	-3.606^{14}	-	-1^{13}	1			
224	42	15 , 6 , 5 , 7 , 6	3^{15}	-3^{20}	3.873^3	-3.873^3				
225	42	16 , 0 , 6 , 21 , 2	4^{13}	-4^8	2^1	-2^{19}	1	0	$D < 0$	Cor. 4.1.2
226	42	16 , 0 , 6 , 21 , 2	4^{12}	-4^9	2^3	-2^{17}	1	0	$D < 0$	Cor. 4.1.2
227	42	16 , 0 , 6 , 21 , 2	4^{11}	-4^{10}	2^5	-2^{15}	1	0	$D < 0$	Cor. 4.1.2
228	42	16 , 0 , 6 , 21 , 2	4^{10}	-4^{11}	2^7	-2^{13}	1			
229	42	16 , 0 , 6 , 21 , 2	4^9	-4^{12}	2^9	-2^{11}	1			
230	42	16 , 0 , 6 , 21 , 2	4^8	-4^{13}	2^{11}	-2^9	1			
231	42	16 , 0 , 6 , 21 , 2	4^7	-4^{14}	2^{13}	-2^7	1	0	$D > D_{max}$	Cor. 4.1.2
232	42	16 , 0 , 6 , 21 , 2	4^6	-4^{15}	2^{15}	-2^5	1	0	$D > D_{max}$	Cor. 4.1.2
233	42	16 , 0 , 6 , 21 , 2	4^5	-4^{16}	2^{17}	-2^3	1	0	$D > D_{max}$	Cor. 4.1.2
234	42	16 , 0 , 6 , 21 , 2	4^4	-4^{17}	2^{19}	-2^1	1	0	$D > D_{max}$	Cor. 4.1.2
235	42	19 , 15 , 8 , 14 , 3	2^{23}	-2^5	5^1	-5^{12}		0	$D < 0$	Cor. 4.1.2
236	42	19 , 15 , 8 , 14 , 3	2^{18}	-2^{10}	5^3	-5^{10}		0	$D < 0$	Cor. 4.1.2
237	42	19 , 15 , 8 , 14 , 3	2^{13}	-2^{15}	5^5	-5^8				
238	42	19 , 15 , 8 , 14 , 3	2^8	-2^{20}	5^7	-5^6				
239	42	19 , 15 , 8 , 14 , 3	2^3	-2^{25}	5^9	-5^4		0	$D > D_{max}$	Cor. 4.1.2
240	42	21 , 12 , 10 , 7 , 6	3^{14}	-3^{21}	4.583^3	-4.583^3				
241	42	23 , 19 , 12 , 14 , 3	2^{22}	-2^6	5^1	-5^{12}		0	$D < 0$	Cor. 4.1.2
242	42	23 , 19 , 12 , 14 , 3	2^{17}	-2^{11}	5^3	-5^{10}		0	$D < 0$	Cor. 4.1.2
243	42	23 , 19 , 12 , 14 , 3	2^{12}	-2^{16}	5^5	-5^8				
244	42	23 , 19 , 12 , 14 , 3	2^7	-2^{21}	5^7	-5^6				
245	42	23 , 19 , 12 , 14 , 3	2^2	-2^{26}	5^9	-5^4		0	$D > D_{max}$	Cor. 4.1.2
246	42	26 , 13 , 16 , 14 , 3	3.606^{14}	-3.606^{14}	-	-2^{13}				
247	42	26 , 22 , 13 , 3 , 14	2^{13}	-2^{26}	11.402^1	-11.402^1				
248	42	26 , 22 , 15 , 7 , 6	2^{11}	-2^{24}	6.782^3	-6.782^3				
249	42	27 , 18 , 17 , 7 , 6	3^{13}	-3^{22}	3.873^3	-3.873^3				
250	42	30 , 26 , 19 , 3 , 14	2^{12}	-2^{27}	10.1^1	-10.1^1				
251	42	32 , 28 , 24 , 14 , 3	2^{17}	-2^{11}	4^1	-4^{12}		0	$D < 0$	Cor. 4.1.2
252	42	32 , 28 , 24 , 14 , 3	2^{15}	-2^{13}	4^2	-4^{11}		0	$D < 0$	Cor. 4.1.2
253	42	32 , 28 , 24 , 14 , 3	2^{13}	-2^{15}	4^3	-4^{10}				
254	42	32 , 28 , 24 , 14 , 3	2^{11}	-2^{17}	4^4	-4^9				
255	42	32 , 28 , 24 , 14 , 3	2^9	-2^{19}	4^5	-4^8				
256	42	32 , 28 , 24 , 14 , 3	2^7	-2^{21}	4^6	-4^7				
257	42	32 , 28 , 24 , 14 , 3	2^5	-2^{23}	4^7	-4^6		0	$D > D_{max}$	Cor. 4.1.2
258	42	32 , 28 , 24 , 14 , 3	2^3	-2^{25}	4^8	-4^5		0	$D > D_{max}$	Cor. 4.1.2
259	42	32 , 28 , 24 , 14 , 3	2^1	-2^{27}	4^9	-4^4		0	$D > D_{max}$	Cor. 4.1.2
260	42	34 , 30 , 27 , 7 , 6	2^9	-2^{26}	4.69^3	-4.69^3				
261	42	37 , 33 , 32 , 2 , 21	2^{12}	-2^{28}	-	-5^1				
262	44	13 , 9 , 2 , 4 , 11	2^{10}	-2^{30}	9^3	-		1	L(K(4,11))	Thm. 5.3.1
263	44	13 , 9 , 2 , 4 , 11	2^{19}	-2^{21}	9^1	-9^2				
264	44	21 , 0 , 10 , 22 , 2	4.583^{11}	-4.583^{11}	-	-1^{21}	1			
265	44	31 , 27 , 20 , 4 , 11	2^{19}	-2^{21}	-	-9^3				
266	44	31 , 27 , 20 , 4 , 11	2^{10}	-2^{30}	9^2	-9^1				
267	45	8 , 4 , 1 , 9 , 5	2^{16}	-2^{20}	4.359^4	-4.359^4				
268	45	16 , 12 , 4 , 5 , 9	2^{16}	-2^{24}	8.718^2	-8.718^2				
269	45	18 , 6 , 7 , 15 , 3	3.464^{15}	-3.464^{15}	3^4	-3^{10}				
270	45	18 , 9 , 6 , 3 , 15	3^{18}	-3^{24}	7.348^1	-7.348^1				
271	45	20 , 16 , 7 , 5 , 9	2^{15}	-2^{25}	9.22^2	-9.22^2				
272	45	22 , 18 , 7 , 3 , 15	2^9	-2^{33}	13^2	-		0	$D > D_{max}$	Cor. 4.1.2
273	45	24 , 15 , 12 , 5 , 9	3^{20}	-3^{20}	-	-6^4				
274	45	24 , 15 , 12 , 5 , 9	3^{18}	-3^{22}	6^1	-6^3				
275	45	24 , 15 , 12 , 5 , 9	3^{16}	-3^{24}	6^2	-6^2				
276	45	24 , 15 , 12 , 5 , 9	3^{14}	-3^{26}	6^3	-6^1				
277	45	24 , 18 , 12 , 9 , 5	2.449^{18}	-2.449^{18}	6^2	-6^6				
278	45	26 , 10 , 15 , 15 , 3	4^{13}	-4^{17}	1^2	-1^{12}				
279	45	26 , 10 , 15 , 15 , 3	4^{12}	-4^{18}	1^6	-1^8				
280	45	26 , 10 , 15 , 15 , 3	4^{11}	-4^{19}	1^{10}	-1^4		0	$D > D_{max}$	Cor. 4.1.2
281	45	28 , 24 , 14 , 3 , 15	2^{14}	-2^{28}	12.41^1	-12.41^1				
282	45	32 , 28 , 20 , 3 , 15	2^{13}	-2^{29}	11.136^1	-11.136^1				
283	45	32 , 28 , 22 , 9 , 5	2^{10}	-2^{26}	5.831^4	-5.831^4				
284	45	38 , 34 , 31 , 3 , 15	2^{15}	-2^{27}	-	-7^2		1	A=T(6)	Thm. 5.3.2
285	45	38 , 34 , 31 , 3 , 15	2^8	-2^{34}	7^2	-		0	$D > D_{max}$	Cor. 4.1.2
286	48	7 , 0 , 1 , 8 , 6	2.646^{20}	-2.646^{20}	-	-1^7	1			
287	48	8 , 4 , 1 , 12 , 4	2^{27}	-2^9	-	-4^{11}		0	$D < 0$	Cor. 4.1.2

Table B.1: continued

Index	v	k , λ_1 , λ_2 , m , n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$	t	#	Notes	Ref.
288	48	8 , 4 , 1 , 12 , 4	2^5	-2^{31}	4^{11}	-		0	$D > D_{max}$	Cor. 4.1.2
289	48	8 , 4 , 1 , 12 , 4	2^{25}	-2^{11}	4^1	-4^{10}		0	$D < 0$	Cor. 4.1.2
290	48	8 , 4 , 1 , 12 , 4	2^{23}	-2^{13}	4^2	-4^9		0	$D < 0$	Cor. 4.1.2
291	48	8 , 4 , 1 , 12 , 4	2^{21}	-2^{15}	4^3	-4^8		0	$D < 0$	Cor. 4.1.2
292	48	8 , 4 , 1 , 12 , 4	2^{19}	-2^{17}	4^4	-4^7		0	$D < 0$	Cor. 4.1.2
293	48	8 , 4 , 1 , 12 , 4	2^{17}	-2^{19}	4^5	-4^6				
294	48	8 , 4 , 1 , 12 , 4	2^{15}	-2^{21}	4^6	-4^5				
295	48	8 , 4 , 1 , 12 , 4	2^{13}	-2^{23}	4^7	-4^4				
296	48	8 , 4 , 1 , 12 , 4	2^{11}	-2^{25}	4^8	-4^3				
297	48	8 , 4 , 1 , 12 , 4	2^9	-2^{27}	4^9	-4^2				
298	48	8 , 4 , 1 , 12 , 4	2^7	-2^{29}	4^{10}	-4^1		0	$D > D_{max}$	Cor. 4.1.2
299	48	10 , 0 , 2 , 16 , 3	3.162^{16}	-3.162^{16}	2^5	-2^{10}	1			
300	48	13 , 4 , 3 , 3 , 16	3^{22}	-3^{23}	-	-5^2				
301	48	14 , 1 , 4 , 16 , 3	3.606^{16}	-3.606^{16}	2^4	-2^{11}				
302	48	14 , 2 , 4 , 12 , 4	3.464^{18}	-3.464^{18}	2^2	-2^9				
303	48	14 , 10 , 1 , 3 , 16	2^{19}	-2^{26}	12.166^1	-12.166^1				
304	48	14 , 10 , 2 , 4 , 12	2^{26}	-2^{18}	-	-10^3		0	$D < 0$	Cor. 4.1.2
305	48	14 , 10 , 2 , 4 , 12	2^{11}	-2^{33}	10^3	-		1	L(K(4,12))	Thm. 5.3.1
306	48	14 , 10 , 2 , 4 , 12	2^{21}	-2^{23}	10^1	-10^2				
307	48	14 , 10 , 2 , 4 , 12	2^{16}	-2^{28}	10^2	-10^1				
308	48	15 , 6 , 3 , 2 , 24	3^{22}	-3^{24}	-	-9^1				
309	48	15 , 6 , 3 , 2 , 24	3^{19}	-3^{27}	9^1	-				
310	48	17 , 1 , 6 , 16 , 3	4^{12}	-4^{20}	1^{15}	-	2			
311	48	17 , 1 , 6 , 16 , 3	4^{15}	-4^{17}	1^3	-1^{12}	2			
312	48	17 , 1 , 6 , 16 , 3	4^{14}	-4^{18}	1^7	-1^8	2			
313	48	17 , 1 , 6 , 16 , 3	4^{13}	-4^{19}	1^{11}	-1^4	2			
314	48	18 , 14 , 3 , 3 , 16	2^{18}	-2^{27}	13.416^1	-13.416^1				
315	48	18 , 14 , 6 , 12 , 4	2^{30}	-2^6	-	-6^{11}		0	$D < 0$	Cor. 4.1.2
316	48	18 , 14 , 6 , 12 , 4	-	-2^{36}	6^{10}	-6^1		0	$D > D_{max}$	Cor. 4.1.2
317	48	18 , 14 , 6 , 12 , 4	2^{27}	-2^9	6^1	-6^{10}		0	$D < 0$	Cor. 4.1.2
318	48	18 , 14 , 6 , 12 , 4	2^{24}	-2^{12}	6^2	-6^9		0	$D < 0$	Cor. 4.1.2
319	48	18 , 14 , 6 , 12 , 4	2^{21}	-2^{15}	6^3	-6^8		0	$D < 0$	Cor. 4.1.2
320	48	18 , 14 , 6 , 12 , 4	2^{18}	-2^{18}	6^4	-6^7				
321	48	18 , 14 , 6 , 12 , 4	2^{15}	-2^{21}	6^5	-6^6				
322	48	18 , 14 , 6 , 12 , 4	2^{12}	-2^{24}	6^6	-6^5				
323	48	18 , 14 , 6 , 12 , 4	2^9	-2^{27}	6^7	-6^4				
324	48	18 , 14 , 6 , 12 , 4	2^6	-2^{30}	6^8	-6^3		0	$D > D_{max}$	Cor. 4.1.2
325	48	18 , 14 , 6 , 12 , 4	2^3	-2^{33}	6^9	-6^2		0	$D > D_{max}$	Cor. 4.1.2
326	48	20 , 10 , 8 , 16 , 3	3.162^{16}	-3.162^{16}	4^5	-4^{10}				
327	48	20 , 12 , 8 , 24 , 2	2.828^{12}	-2.828^{12}	4^9	-4^{14}				
328	48	21 , 6 , 9 , 24 , 2	3.873^{12}	-3.873^{12}	3^8	-3^{15}				
329	48	21 , 8 , 9 , 12 , 4	3.606^{18}	-3.606^{18}	3^2	-3^9				
330	48	22 , 2 , 10 , 24 , 2	4.472^{12}	-4.472^{12}	2^6	-2^{17}				
331	48	22 , 6 , 10 , 16 , 3	4^{17}	-4^{15}	-	-2^{15}	2	0	$D < 0$	Cor. 4.1.2
332	48	22 , 6 , 10 , 16 , 3	4^{16}	-4^{16}	2^2	-2^{13}	2			
333	48	22 , 6 , 10 , 16 , 3	4^{15}	-4^{17}	2^4	-2^{11}	2			
334	48	22 , 6 , 10 , 16 , 3	4^{14}	-4^{18}	2^6	-2^9	2			
335	48	22 , 6 , 10 , 16 , 3	4^{13}	-4^{19}	2^8	-2^7	2			
336	48	22 , 6 , 10 , 16 , 3	4^{12}	-4^{20}	2^{10}	-2^5	2			
337	48	22 , 6 , 10 , 16 , 3	4^{11}	-4^{21}	2^{12}	-2^3	2	0	$D > D_{max}$	Cor. 4.1.2
338	48	22 , 6 , 10 , 16 , 3	4^{10}	-4^{22}	2^{14}	-2^1	2	0	$D > D_{max}$	Cor. 4.1.2
339	48	22 , 18 , 6 , 3 , 16	2^{24}	-2^{21}	-	-14^2		0	$D < 0$	Cor. 4.1.2
340	48	22 , 18 , 6 , 3 , 16	2^{10}	-2^{35}	14^2	-		0	$D > D_{max}$	Cor. 4.1.2
341	48	22 , 18 , 6 , 3 , 16	2^{17}	-2^{28}	14^1	-14^1				
342	48	23 , 0 , 11 , 24 , 2	4.796^{12}	-4.796^{12}	-	-1^{23}	1			
343	48	24 , 8 , 12 , 12 , 4	4^{15}	-4^{21}	0^{11}	-	2,3			
344	48	26 , 6 , 14 , 24 , 2	4.472^{12}	-4.472^{12}	2^5	-2^{18}				
345	48	26 , 10 , 14 , 16 , 3	4^9	-4^{23}	2^{15}	-	2	0	$D > D_{max}$	Cor. 4.1.2
346	48	26 , 10 , 14 , 16 , 3	4^{16}	-4^{16}	2^1	-2^{14}	2			
347	48	26 , 10 , 14 , 16 , 3	4^{15}	-4^{17}	2^3	-2^{12}	2			
348	48	26 , 10 , 14 , 16 , 3	4^{14}	-4^{18}	2^5	-2^{10}	2			
349	48	26 , 10 , 14 , 16 , 3	4^{13}	-4^{19}	2^7	-2^8	2			
350	48	26 , 10 , 14 , 16 , 3	4^{12}	-4^{20}	2^9	-2^6	2			
351	48	26 , 10 , 14 , 16 , 3	4^{11}	-4^{21}	2^{11}	-2^4	2	0	$D > D_{max}$	Cor. 4.1.2
352	48	26 , 10 , 14 , 16 , 3	4^{10}	-4^{22}	2^{13}	-2^2	2	0	$D > D_{max}$	Cor. 4.1.2
353	48	26 , 22 , 10 , 3 , 16	2^{23}	-2^{22}	-	-14^2		0	$D < 0$	Cor. 4.1.2
354	48	26 , 22 , 10 , 3 , 16	2^9	-2^{36}	14^2	-		0	$D > D_{max}$	Cor. 4.1.2
355	48	26 , 22 , 10 , 3 , 16	2^{16}	-2^{29}	14^1	-14^1				
356	48	27 , 12 , 15 , 24 , 2	3.873^{12}	-3.873^{12}	3^7	-3^{16}				
357	48	27 , 14 , 15 , 12 , 4	3.606^{18}	-3.606^{18}	3^1	-3^{10}				
358	48	28 , 18 , 16 , 16 , 3	3.162^{16}	-3.162^{16}	4^4	-4^{11}				
359	48	28 , 20 , 16 , 24 , 2	2.828^{12}	-2.828^{12}	4^8	-4^{15}				
360	48	30 , 26 , 15 , 3 , 16	2^{15}	-2^{30}	13.416^1	-13.416^1				
361	48	30 , 26 , 18 , 12 , 4	2^{27}	-2^9	-	-6^{11}		0	$D < 0$	Cor. 4.1.2
362	48	30 , 26 , 18 , 12 , 4	-	-2^{36}	6^9	-6^2		0	$D > D_{max}$	Cor. 4.1.2
363	48	30 , 26 , 18 , 12 , 4	2^{24}	-2^{12}	6^1	-6^{10}		0	$D < 0$	Cor. 4.1.2
364	48	30 , 26 , 18 , 12 , 4	2^{21}	-2^{15}	6^2	-6^9		0	$D < 0$	Cor. 4.1.2
365	48	30 , 26 , 18 , 12 , 4	2^{18}	-2^{18}	6^3	-6^8				
366	48	30 , 26 , 18 , 12 , 4	2^{15}	-2^{21}	6^4	-6^7				
367	48	30 , 26 , 18 , 12 , 4	2^{12}	-2^{24}	6^5	-6^6				
368	48	30 , 26 , 18 , 12 , 4	2^9	-2^{27}	6^6	-6^5				
369	48	30 , 26 , 18 , 12 , 4	2^6	-2^{30}	6^7	-6^4		0	$D > D_{max}$	Cor. 4.1.2
370	48	30 , 26 , 18 , 12 , 4	2^3	-2^{33}	6^8	-6^3		0	$D > D_{max}$	Cor. 4.1.2
371	48	31 , 15 , 20 , 16 , 3	4^{14}	-4^{18}	-	-1^{15}	2			
372	48	31 , 15 , 20 , 16 , 3	4^{13}	-4^{19}	1^4	-1^{11}	2			
373	48	31 , 15 , 20 , 16 , 3	4^{12}	-4^{20}	1^8	-1^7	2			
374	48	31 , 15 , 20 , 16 , 3	4^{11}	-4^{21}	1^{12}	-1^3	2	0	$D > D_{max}$	Cor. 4.1.2
375	48	33 , 24 , 21 , 2 , 24	3^{19}	-3^{27}	-	-9^1				
376	48	33 , 24 , 21 , 2 , 24	3^{16}	-3^{30}	9^1	-				

Table B.1: continued

Index	v	k	λ_1	λ_2	m	n	$\alpha_1^{f_1}$	$\alpha_2^{f_2}$	$\alpha_3^{g_1}$	$\alpha_4^{g_2}$	t	#	Notes	Ref.
377	48	34	30	21	3	16	2^{14}	-2^{31}	12.166^{1}	-12.166^{1}				
378	48	34	30	22	4	12	2^{21}	-2^{23}	-	-10^{3}				
379	48	34	30	22	4	12	2^{6}	-2^{38}	10^{3}	-		0	$D > D_{max}$	Cor. 4.1.2
380	48	34	30	22	4	12	2^{16}	-2^{28}	10^{1}	-10^{2}				
381	48	34	30	22	4	12	2^{11}	-2^{33}	10^{2}	-10^{1}				
382	48	35	26	25	3	16	3^{15}	-3^{30}	5^{2}	-				
383	48	35	28	25	8	6	2.646^{20}	-2.646^{20}	-	-5^{7}				
384	48	35	30	25	12	4	2.236^{18}	-2.236^{18}	5^{2}	-5^{9}				
385	48	38	34	28	3	16	2^{18}	-2^{27}	-	-10^{2}		2	A=Shrikhande	Thm. 5.3.2
386	48	38	34	28	3	16	2^{8}	-2^{37}	10^{2}	-		0	$D > D_{max}$	Cor. 4.1.2
387	48	38	34	28	3	16	2^{13}	-2^{32}	10^{1}	-10^{1}				
388	48	40	36	33	12	4	2^{19}	-2^{17}	-	-4^{11}		0	$D < 0$	Cor. 4.1.2
389	48	40	36	33	12	4	2^{17}	-2^{19}	4^{1}	-4^{10}				
390	48	40	36	33	12	4	2^{15}	-2^{21}	4^{2}	-4^{9}				
391	48	40	36	33	12	4	2^{13}	-2^{23}	4^{3}	-4^{8}				
392	48	40	36	33	12	4	2^{11}	-2^{25}	4^{4}	-4^{7}				
393	48	40	36	33	12	4	2^{9}	-2^{27}	4^{5}	-4^{6}				
394	48	40	36	33	12	4	2^{7}	-2^{29}	4^{6}	-4^{5}		0	$D > D_{max}$	Cor. 4.1.2
395	48	40	36	33	12	4	2^{5}	-2^{31}	4^{7}	-4^{4}		0	$D > D_{max}$	Cor. 4.1.2
396	48	40	36	33	12	4	2^{3}	-2^{33}	4^{8}	-4^{3}		0	$D > D_{max}$	Cor. 4.1.2
397	48	40	36	33	12	4	2^{1}	-2^{35}	4^{9}	-4^{2}		0	$D > D_{max}$	Cor. 4.1.2
398	48	42	38	36	3	16	2^{15}	-2^{30}	-	-6^{2}		1	A=Clebsch	Thm. 5.3.2
399	48	42	38	36	3	16	2^{9}	-2^{36}	6^{2}	-		0	$D > D_{max}$	Cor. 4.1.2
400	48	42	38	36	3	16	2^{12}	-2^{33}	6^{1}	-6^{1}				
401	49	12	8	2	7	7	2^{18}	-2^{24}	6.782^{3}	-6.782^{3}				
402	49	16	12	4	7	7	2^{17}	-2^{25}	7.746^{3}	-7.746^{3}				
403	49	18	9	6	7	7	3^{18}	-3^{24}	5.477^{3}	-5.477^{3}				
404	49	24	15	11	7	7	3^{17}	-3^{25}	6.083^{3}	-6.083^{3}				
405	49	28	21	15	7	7	2.646^{21}	-2.646^{21}	7^{1}	-7^{5}				
406	49	40	36	32	7	7	2^{11}	-2^{31}	5.657^{3}	-5.657^{3}				
407	50	25	15	12	10	5	3.162^{20}	-3.162^{20}	5^{2}	-5^{7}				
408	50	25	24	12	25	2	-	-1^{25}	5^{12}	-5^{12}		15		Thm. 5.2.4
409	50	33	24	21	5	10	3^{17}	-3^{28}	6.245^{2}	-6.245^{2}				

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